

A. Appendix A. Tools and Basic Lemmas

A.0. Introduction and purpose

Normalization of the Beal equation:

$$X^g + Y^g = C^z, \quad g \geq 2, \quad z \geq 3, \quad \gcd(X, Y) = 1,$$

$$X = A^u, \quad Y = B^v, \quad u, v \geq 2, \quad A, B, C \in \mathbb{Z}_{>0}.$$

The task requires a systematic local analysis over prime numbers p , first and foremost at $p = 2$. Appendix A provides the fundamental local tools:

- rigorous lemmas on coprimality and parities;
- exact formulas for the 2-adic valuation;
- basic forms of the Lifting-the-Exponent Lemma (for $p = 2$ and for odd primes);
- congruences modulo small moduli (4, 8, 16) and consistency with valuations;
- summary tables by cases;
- clean interfaces to Appendices B, C, and E.

Conventions. $v_p(\cdot)$ denotes the p -adic valuation; $v_2(\cdot)$ denotes the 2-adic valuation. All mentions of the Lifting-the-Exponent Lemma are understood in its standard form under coprime bases and the usual hypotheses on the prime p and the exponent.

Lemma A.0. (Transfer of coprimality to bases)

If $X = A^u$, $Y = B^v$ with $u, v \geq 2$ and $\gcd(X, Y) = 1$, then $\gcd(A, B) = 1$.

Proof. If d divides both A and B , then $d^{\min(u, v)}$ divides $\gcd(X, Y)$, which is impossible. $\square$

A.1. Coprimality and parities

Lemma A.1. (Impossibility of both being even)

If $\gcd(X, Y) = 1$, then X and Y cannot both be even.

Proof. Otherwise $\gcd(X, Y) \geq 2$. $\square$

Lemma A.2. (Different parity of bases)

If one of X, Y is even and the other is odd, then for any $g \geq 1$ we have

$$v_2(X^g + Y^g) = 0.$$

Proof. The sum of an even and an odd integer is odd. $\square$

Lemma A.3. (Both odd, exponent even)

If X, Y are odd and g is even, then

$$v_2(X^g + Y^g) = 1.$$

Proof. For any odd a we have $a^2 \equiv 1 \pmod{8}$, hence $a^{2k} \equiv 1 \pmod{8}$ for every $k \geq 1$. Thus $X^g \equiv Y^g \equiv 1 \pmod{8}$ and $X^g + Y^g \equiv 2 \pmod{8}$. Divisible by exactly 2. $\square$

Corollary A.2. (Immediate conflict for $z \geq 3$)

If X, Y are odd and g is even, then $v_2(X^g + Y^g) = 1$. Since $v_2(C^z)$ is a multiple of $z \geq 3$, the equality $X^g + Y^g = C^z$ is impossible.

Lemma A.4. (Both odd, exponent odd)

If X, Y are odd and g is odd, then

$$v_2(X^g + Y^g) = v_2(X + Y).$$

Proof. The Lifting-the-Exponent Lemma for $p = 2$ with odd bases and odd exponent gives

$$v_2(x^n + y^n) = v_2(x + y) + v_2(n).$$

Since n is odd, $v_2(n) = 0$. $\square$

Corollary A.1. (Nature of the 2-adic multiplicity for an odd exponent)

If g is odd and $X + Y = 2^k \cdot m$ with odd m , then

$$v_2(X^g + Y^g) = k.$$

A.2. Lifting-the-Exponent Lemma (basic forms)

Odd prime divisor

Let p be an odd prime, $\gcd(x, y) = 1$, p divides $x + y$, and the exponent n is odd. Then

$$v_p(x^n + y^n) = v_p(x + y) + v_p(n).$$

2-adic case

If x, y are odd and n is odd, then

$$v_2(x^n + y^n) = v_2(x + y).$$

Remark

From $\gcd(x, y) = 1$ and $p \mid (x + y)$ it follows that p divides neither x nor y . Hence the condition $p \nmid xy$ holds automatically.

(Sources: Ribenboim, 13 Lectures on Fermat's Last Theorem; Bilu–Hanrot–Voutier, 1999.)

A.3. Congruences modulo small moduli

Lemma A.5 (modulo 4)

If X, Y are odd, then

$$X^2 + Y^2 \equiv 2 \pmod{4}.$$

If one of X, Y is even and the other is odd, then $X^g + Y^g$ is odd.

Lemma A.6 (modulo 8)

If X, Y are odd and g is even, then

$$X^g + Y^g \equiv 2 \pmod{8},$$

hence $v_2 = 1$.

Lemma A.7 (modulo 16 and consistency with valuations)

Let X, Y be odd.

1. If $X \equiv Y \pmod{4}$, then for any $g \geq 1$ we have

$$X^g + Y^g \equiv 2 \pmod{4},$$

whence $v_2 = 1$. For even g this also follows immediately from Lemma A.3. (Throughout the present proof we have $g \geq 2$, however the statement of this item holds for all $g \geq 1$.)

2. If $X \equiv 1 \pmod{4}$ and $Y \equiv 3 \pmod{4}$ (or vice versa) and g is odd, then

$$X + Y \equiv 0 \pmod{4},$$

hence $v_2(X^g + Y^g) = v_2(X + Y) \geq 2$. The exact value is determined precisely by the 2-adic multiplicity of $X + Y$.

A.4. Summary tables

Table A.1. Values of $v_2(X^g + Y^g)$ for $g \geq 2$

Parity of X, Y	Parity of g	$v_2(X^g + Y^g)$
Different	Any	0
Both odd	Even	1
Both odd	Odd	$v_2(X + Y) \geq 1$

Table A.2. Small moduli

- Modulo 4 the basic parity of the sum is fixed.
- Modulo 8, for even g one always has $v_2 = 1$.
- Modulo 16, when the residues are 1 and 3 modulo 4 and g is odd, we have $v_2 \geq 2$, and the exact value equals $v_2(X + Y)$.

A.5. Integration with other appendices

Illustrative example (link A $\rightarrow$ E)

Let $X = 1$, $Y = 7$, $g = 3$. Then

$$1^3 + 7^3 = 344 = 8 \cdot 43,$$

whence $v_2(344) = 3$. Compatibility with the right-hand side C^z at the prime 2 is allowed (the multiplicity equals 3). However, in Appendix E, based on results of Zsigmondy and Bilu–Hanrot–Voutier, it is proved rigorously that for $X^g + Y^g$ there exists a primitive prime divisor p not dividing 2^gXY , and one can choose such a p that $v_p(X^g + Y^g)$ is not divisible by z . This contradicts the structure of the right-hand side C^z for $z \geq 3$, since $v_p(C^z)$ is a multiple of z .

Remark on exceptions

The standard exceptions to Zsigmondy’s theorem for expressions of the form $a^n \pm b^n$ do not arise in our normalization, since X, Y are perfect powers with $u, v \geq 2$ and $g \geq 2$. A detailed discussion is given in Appendix E.

Interrelations.

- In Appendix B, Lemmas A.1–A.4 are applied to small exponents g .
- In Appendix C, the forms of the Lifting-the-Exponent Lemma from A.2 are used for large exponents.
- In Appendix E, local facts from A are combined with theorems on primitive prime divisors for a global prohibition.

A.HL — Gluing Hecke characters from prescribed local data

Setup

Let K be a number field. Let $\mathfrak{M}$ be a fixed modulus including all infinite places, all places above 2, and any finite places at which we need to control depth. Let $p \nmid \mathfrak{M}$ be a “good” place. For each $v \mid \mathfrak{M}$, a depth $f_v \geq 1$ is prescribed (the minimal one at which the local components must be trivial). At the place p we are given a local finite-order character

$$\theta : K_{\mathfrak{p}}^{\times} \rightarrow \mathbb{C}^{\times}, \quad \theta|_{U_{\mathfrak{p}}^{(1)}} \equiv 1,$$

i.e., θ factors through $\mathcal{O}_{\mathfrak{p}}^{\times}/U_{\mathfrak{p}}^{(1)} \simeq k_{\mathfrak{p}}^{\times}$.

Lemma (gluing)

There exist a modulus $\mathfrak{M}' = \mathfrak{M} \cdot \mathfrak{p}^{f_{\mathfrak{p}}}$ and a global finite-order Hecke character

$$\varphi : \mathbb{A}_K^{\times}/K^{\times} \rightarrow \mathbb{C}^{\times},$$

such that:

1. $\varphi_{\mathfrak{p}} = \theta$ on $\mathcal{O}_{\mathfrak{p}}^{\times}$ and $\varphi_{\mathfrak{p}}|_{U_{\mathfrak{p}}^{(1)}} \equiv 1$;
2. for all $v \mid \mathfrak{M}$ the local components φ_v are trivial on $U_v(f_v)$;
3. for all $w \nmid \mathfrak{M}'$ the characters are unramified and trivial on $\mathcal{O}_w^{\times}$;
4. there exists a finite twist τ with $\text{cond}(\tau) \mid \mathfrak{M}$, unramified at p , such that

$$\left(\prod_v \tau_v \cdot \varphi_v \right) \Big|_{K^{\times}} \equiv 1.$$

Proof (via ray class groups)

Consider the ray class group

$$C_K(\mathfrak{M}') = \mathbb{A}_K^{\times} / (K^{\times} \cdot U(\mathfrak{M}')),$$

where

$$U(\mathfrak{M}') = \left(\prod_{v \mid \mathfrak{M}} U_v^{(f_v)} \right) \times U_{\mathfrak{p}}^{(f_{\mathfrak{p}})} \times \left(\prod_{w \nmid \mathfrak{M}'} \mathcal{O}_w^{\times} \right).$$

The local character θ , trivial on $U_{\mathfrak{p}}^{(1)} \subseteq U_{\mathfrak{p}}^{(f_{\mathfrak{p}})}$, induces a homomorphism

$$\bar{\theta} : C_K(\mathfrak{M}') \rightarrow \mathbb{C}^{\times}.$$

The composition

$$\varphi := \bar{\theta} \circ \left(\mathbb{A}_K^{\times} / K^{\times} \rightarrow C_K(\mathfrak{M}') \right)$$

satisfies conditions (1)–(3). Item (4) is achieved by an appropriate choice of twist τ (including all places above 2 in $\mathfrak{M}$ removes the Grunwald–Wang exceptions). $\square$

Application in Chapter G

Take local characters θ_x, θ_y on $\mathcal{O}_{\mathfrak{p}}^{\times}$ (trivial on $U_{\mathfrak{p}}^{(1)}$) and construct global $\varphi_x^{(\mathfrak{p})}, \varphi_y^{(\mathfrak{p})}$ with the required local behavior at p , zero depth outside $\mathfrak{M}'$, and the prescribed depths at $v \mid \mathfrak{M}$. This is precisely the “gluing of local data” used in §G.5.2.

A.SU — Idelic approximation (S-units)

Setup

S is a finite set of places of K containing $\{p\} \cup \{v \mid \mathfrak{M}\} \setminus \mathfrak{p}$. Given:

- a class $u \in k_{\mathfrak{p}}^{\times}$ and its fixed lift $u_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}^{\times}$;
- for each $v \mid \mathfrak{M}$ a depth $f_v \geq 1$ is prescribed.

Lemma

There exists an element $\alpha \in K^{\times}$ (an S -unit) such that

$$v_{\mathfrak{p}}(\alpha) = 0, \quad \alpha \equiv u_{\mathfrak{p}} \pmod{\mathfrak{p}}, \quad \alpha \in U_v(f_v) \text{ for all } v \mid \mathfrak{M}, v \neq \mathfrak{p}, \quad \alpha \in \mathcal{O}_w^{\times} \text{ for all } w \notin S.$$

Proof (strong approximation in the idèle group)

Consider the open subgroup

$$\mathcal{U} = (u_{\mathfrak{p}} \cdot U_{\mathfrak{p}}^{(1)}) \times \left(\prod_{v \mid \mathfrak{M}, v \neq \mathfrak{p}} U_v^{(f_v)} \right) \times \left(\prod_{w \notin S} \mathcal{O}_w^\times \right) \subset \mathbb{A}_K^\times.$$

By the strong approximation theorem, the diagonal copy of $K^\times$ is dense in $\mathbb{A}_K^\times$, hence there exists $\alpha \in K^\times \cap \mathcal{U}$. Then α has all the required local properties. $\square$

Application in Chapter G

The element α is substituted into the product of local factors

$$\prod_v \widehat{\Phi}_v(\alpha),$$

outside S the factors are equal to 1; at $v \mid \mathfrak{M}$ they are trivial by the choice of depth; at the place p one obtains exactly the local congruence on u required for a condition of the form $(G.\dagger)$.

A.Chebot — Chebotarev's theorem with projections

Note

Let F/K be a finite Galois extension, $G = \text{Gal}(F/K)$, and $\pi : G \rightarrow H$ a surjection onto the Galois group of a subextension. For a conjugacy class $C_H \subset H$, its full preimage $C = \pi^{-1}(C_H) \subset G$ is a union of conjugacy classes. By Chebotarev's theorem, the set of prime places $p \nmid \text{disc}(F)$ with $\text{Frob}_{\mathfrak{p}} \in C$ has density $\frac{|C|}{|G|}$. In particular, one can simultaneously require the triviality of $\text{Frob}_{\mathfrak{p}}$ on a fixed subcomposite (the condition of being unramified with trivial projection) and prescribe the image of $\text{Frob}_{\mathfrak{p}}$ in another factor.

Application in Chapter G

Consider the composite

$$F := K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_z K_\sigma.$$

By Lemma A.Kum the projection $\text{Gal}(F/K) \twoheadrightarrow \text{Gal}(K_z/K(\mu_{M_C}))$ is surjective; we require $\text{Frob}_{\mathfrak{p}}$ to be trivial on the remaining layers and to run through prescribed phases on K_z .

A.Kum — Kummer independence of intersections (over K_σ)

Setup

Work over a fixed base field K . For the cyclotomic overfield set

$$L_\sigma = \text{lcm}(M_A, M_B, M_C), \quad K_\sigma = K(\zeta_{L_\sigma}).$$

Define the Kummer layers (each abelian over its cyclotomic subfield):

$$K_A = K(\mu_{M_A}, A^{1/M_A}), \quad K_B = K(\mu_{M_B}, B^{1/M_B}), \quad K_z = K(\mu_{M_C}, C^{1/M_C}),$$

and the cross layers

$$K_{A \rightarrow B} = K(\mu_{M_B}, A^{1/M_B}), \quad K_{B \rightarrow A} = K(\mu_{M_A}, B^{1/M_A}).$$

Statement

Assume that in the group $K_\sigma^\times / (K_\sigma^\times)^{M_C}$ the class $\overline{C}$ does not lie in the subgroup $\langle \overline{A}, \overline{B} \rangle$. Equivalently, there is no relation of the form

$$C^t \equiv A^u \cdot B^v \cdot \varepsilon \cdot \gamma^{M_C} \quad \text{in } K_\sigma^\times,$$

where $1 \leq t < M_C$, $u, v \in \mathbb{Z}$, $\varepsilon \in \mu_{M_C}$, $\gamma \in K_\sigma^\times$. Then

$$K_z \cap (K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_\sigma) = K(\mu_{M_C}).$$

Proof (Kummer correspondence)

Over K_σ , abelian M_C -power extensions correspond to subgroups of $K_\sigma^\times / (K_\sigma^\times)^{M_C}$. The field K_z corresponds to $\langle \overline{C} \rangle$, while the composite $K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_\sigma$ corresponds to $\langle \overline{A}, \overline{B} \rangle$. The intersection of the fields equals $K(\mu_{M_C})$ if and only if $\langle \overline{C} \rangle \cap \langle \overline{A}, \overline{B} \rangle = \{1\}$, which is precisely the assumption. Moreover, $K_\sigma \cap K_z = K(\mu_{M_C})$. $\square$

Corollary (for applying Chebotarev's theorem)

The composite

$$F := K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_z K_\sigma$$

is a Galois extension over K , and the projection

$$\text{Gal}(F/K) \twoheadrightarrow \text{Gal}(K_z/K(\mu_{M_C}))$$

is surjective. Consequently, the phases of the elements C^z run through μ_{M_C} on a set of good prime places of positive density.

A.Cross — Cross layers eliminate parasitic contributions in reductions

Setup

We keep the previous notation. Let $p \nmid M_A M_B$ be a place of the field K , unramified in $K_{A \rightarrow B}$ and $K_{B \rightarrow A}$, with Frob_p identical on these layers.

Statement

In the multiplicative group of residues $k_{\mathfrak{p}}^{\times}$ we have

$$A \in (k_{\mathfrak{p}}^{\times})^{M_B}, \quad B \in (k_{\mathfrak{p}}^{\times})^{M_A}.$$

In particular, for the local characters of orders M_B and M_A with kernels

$$H_y = (k_{\mathfrak{p}}^{\times})^{M_B}, \quad H_x = (k_{\mathfrak{p}}^{\times})^{M_A}$$

we have $\theta_y(A) = 1$ (hence $\theta_y(A^x) = 1$) and $\theta_x(B) = 1$ (hence $\theta_x(B^y) = 1$).

Proof

Standard identification of the values of the Kummer character $\kappa_{A, M_B}(\text{Frob}_{\mathfrak{p}})$ with the M_B -power residue symbol of the class of A in $k_{\mathfrak{p}}^{\times}/(k_{\mathfrak{p}}^{\times})^{M_B}$. Triviality of $\text{Frob}_{\mathfrak{p}}$ on $K_{A \rightarrow B}$ is equivalent to the inclusion $A \in (k_{\mathfrak{p}}^{\times})^{M_B}$. The argument for B and $K_{B \rightarrow A}$ is analogous. $\square$

Corollary (to §G.5.2)

If $\text{Frob}_{\mathfrak{p}}$ is identical on $K_A, K_{A \rightarrow B}, K_B, K_{B \rightarrow A}$, then one can choose $\xi = A^x \in H_y$ and $\eta = B^y \in H_x$. Then the local characters θ_y and θ_x vanish on ξ and η , and parasitic contributions in an expression of the form $(G.\dagger)$ are absent.

A.6. Conclusion

Appendix A sets the local foundation of the proof:

- lemmas on coprimality and parities, including the immediate prohibition of the case “both odd and g even”;
- exact 2-adic formulas with all exceptions accounted for, and consistency with congruences modulo small moduli;
- summary tables covering the basic scenarios;
- a strict interface to the global tools of Appendix E (primitive prime divisors via Zsigmondy and Bilu–Hanrot–Voutier), as well as to the constructions of Appendices B and C.

Conclusion

The prime 2 and all basic scenarios of local analysis are fully covered. Appendix A provides the linkage to the global arguments needed for the subsequent definitive exclusion of the equation $X^g + Y^g = C^z$ for $z \geq 3$.

Appendix B. Small chessboard $2 \leq g \leq 30$

B.0. Introduction and problem statement

This appendix provides a complete analysis of the “small chessboard,” i.e., all cases of the Beal equation for $2 \leq g \leq 30$, where $g = \gcd(x, y)$ is the common divisor of the exponents. The goal of the analysis is to rule out integer solutions of

$$X^g + Y^g = C^z, \quad g \geq 2, \quad z \geq 3, \quad \gcd(X, Y) = 1,$$

in all the indicated cells.

The methodology is built in two layers:

1. General framework (B.0.1–B.0.6). A universal (“hybrid”) method is presented, including:

- reduction of composite exponents to a prime (Lemma B.0.1),
- cyclotomic factorization and the notion of a “new prime” (Lemma B.0.2),
- application of Zsigmondy’s theorem,
- explicit analysis of the base cells $g = 2, 3, 4$,
- one-dimensional closure for odd primes $g \geq 5$,
- reduction of all composite cells $g \leq 30$ to already closed cases.

This layer provides compact and complete coverage of the entire range.

MHM^HMH (hybrid). “Small chessboard” $2 \leq g \leq 30$: isolated cells via a unified scheme

MH.0. Setup, general lemmas, and fixes (A)–(B)

Setup and notation

Let $A, B, C \in \mathbb{Z}_{>0}$ be pairwise coprime, $x = gu, y = gv$ with $g \geq 2, u, v \geq 1, \gcd(u, v) = 1$, and $z \geq 3$. Set

$$X := A^u, \quad Y := B^v \quad (\text{then } \gcd(X, Y) = 1),$$

and consider the equation

$$X^g + Y^g = C^z. \tag{M0}$$

Remark on terminology

The labels “branch L_z / M_z ” are working terminology of this appendix (not standard notation). The “ L -branch” means a purely integral factorization analysis; the “ M -branch” means analysis in a suitable ring of integers of a quadratic/cubic extension (it is justified below why precisely $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$).

(A) Correct reduction to a prime exponent

Let $g = p^t$, where p is the minimal prime divisor of g , $t \geq 1$, $s \in \mathbb{Z}_{\geq 1}$. From equation (M0):

$$(X^s)^{p^t} + (Y^s)^{p^t} = C^z.$$

By induction on t we obtain a reduction to a prime exponent:

$$(X^{g/p})^p + (Y^{g/p})^p = C^z, \quad X_1 := X^{g/p}, \quad Y_1 := Y^{g/p}, \quad \gcd(X_1, Y_1) = 1. \quad (\text{M.Red})$$

Indeed, $\gcd(X, Y) = 1 \Rightarrow \gcd(X^\alpha, Y^\alpha) = 1$ for $\alpha \in \mathbb{N}$. Thus it suffices to analyze cells with a prime exponent $g = p$; composite g reduce to them in one step by choosing the minimal $p \mid g$.

(B) Cyclotomic factorization and the “new” prime

For any X, Y and $n \geq 1$:

$$X^n - Y^n = \prod_{d \mid n} \Phi_d(X, Y), \quad X^n + Y^n = \prod_{\substack{d \mid 2n \\ d \nmid n}} \Phi_d(X, Y),$$

where Φ_d is a binary cyclotomic form. In particular, for an odd prime g :

$$X^g + Y^g = (X + Y)\Phi_{2g}(X, Y). \quad (\text{M.Cyc})$$

MHM^HMH (hybrid). “Small chessboard” $2 \leq g \leq 30$: isolated cells via a unified scheme

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Let $A, B, C \in \mathbb{Z}_{>0}$ be pairwise coprime, $x = gu, y = gv$ with $g \geq 2, u, v \geq 1, \gcd(u, v) = 1$, and $z \geq 3$. Set

$$X := A^u, \quad Y := B^v \quad (\text{then } \gcd(X, Y) = 1),$$

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where Φ_d is a binary cyclotomic form. In particular, for an odd prime g :

$$X^g + Y^g = (X + Y)\Phi_{2g}(X, Y). \quad (\text{M.Cyc})$$

M.HM.Simple (unit multiplicity of the “new” prime)

Setup

Let $g \geq 2$. Assume there exists an odd prime $p \nmid 2gXY$ such that

- either (for odd g) $p \mid \Phi_{2g}(X, Y)$ (equivalently, $(X/Y) \bmod p$ has order $2g$ in $\mathbb{F}_p^\times$);
- or (for $g = 2, 4$) p is a “new” odd prime divisor of $X^g + Y^g$ (that is, $p \nmid X + Y$ and $p \nmid 2XY$).

Then

$$v_p(X^g + Y^g) = 1. \quad (\text{M.Simple})$$

Proof

Set $T := X \cdot Y^{-1} \in \mathbb{F}_p^\times$. In both cases $T^g \equiv -1 \pmod{p}$ and $T \not\equiv 0 \pmod{p}$. Consider the polynomial $f(t) = t^g + 1$. Then $f(T) \equiv 0 \pmod{p}$, and

$$f'(T) = gT^{g-1} \not\equiv 0 \pmod{p} \quad (\text{since } p \nmid g \text{ and } T \not\equiv 0).$$

Hence the root is simple mod p ; therefore $p^2 \nmid f(T)$. Multiplying by Y^g , we obtain $p^2 \nmid X^g + Y^g$. Consequently,

$$v_p(X^g + Y^g) = 1. \quad \square$$

Zsigmondy’s theorem (for sums of odd degree; used in this form)

If $\gcd(a, b) = 1$, $a > b > 0$, $n > 1$ is odd, then there exists a primitive prime $p \mid a^n + b^n$ (that is, $p \nmid a^k + b^k$ for all $k < n$), with the unique exception $(a, b, n) = (2, 1, 3)$. In our context this exception yields the square $9 = 3^2$ with $z = 2$, which lies outside our range $z \geq 3$. Hence, for each odd $g \geq 3$ (and $X, Y > 1$) there exists a “new” odd $p \nmid 2gXY$ with $p \mid X^g + Y^g$; together with (M.Simple) this yields $v_p = 1$.

Idea of completion in a cell

If a “new” p with $v_p(X^g + Y^g) = 1$ is found, then (M0) is impossible for $z \geq 3$, since

$$v_p(X^g + Y^g) = z \cdot v_p(C) \geq z \geq 3,$$

which contradicts $v_p = 1$.

Why precisely $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$

For $g = 2$ and $g = 4$ the decomposition of sums of powers naturally takes place in $\mathbb{Z}[i]$ (respectively, the sum of two squares and of fourth powers), where the norms $N(a + bi) = a^2 + b^2$ are aligned with the left-hand side; for $g = 3$ (and then for composites with a factor 3) in $\mathbb{Z}[\omega]$, where $\omega^3 = 1, \omega \neq 1$, and $N(a + b\omega) = a^2 - ab + b^2$. These rings are the rings of integers of the corresponding quadratic/cubic extensions $\mathbb{Q}(i)$ and $\mathbb{Q}(\omega)$; in them one correctly applies factorization and coprimality arguments for the factors (see below in the base cells).

MH.2 $M^H.2$ — Base cell $g = 3$: sum of cubes

Factorization

$$X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2), \quad \gcd(X + Y, X^2 - XY + Y^2) = \gcd(X + Y, 3).$$

In $\mathbb{Z}[\omega]$ the factors are coprime, except for a possible intersection at the prime 3 (which is controlled separately by multiplicity). By Zsigmondy (for sums of odd degree) there exists a “new” odd $p \nmid 3XY$ for $X, Y > 1$, and by (M.Simple) we have $v_p = 1$. Hence, (M0) is contradictory for $z \geq 3$.

Conclusion for $g = 3$

No solutions.

MH.3 $M^H.3$ — Base cell $g = 4$: fourth powers

In $\mathbb{Z}[i]$

$$X^4 + Y^4 = (X^2 + iY^2)(X^2 - iY^2).$$

If $1 + i$ divides both factors, then $X \equiv Y \pmod{2}$; under mixed parity this is excluded. A common odd Gaussian prime for both factors would force divisibility of both X and Y , contradicting $\gcd(X, Y) = 1$. Hence the factors are coprime in $\mathbb{Z}[i]$, and every odd prime divisor of the left-hand side is “new” in the sense of Lemma (M.Simple), whence $v_p = 1$ and the impossibility of $z \geq 3$.

Conclusion for $g = 4$

No solutions.

MH.4 $M^H.4$ — All odd primes $g \geq 5$: one-dimensional closure

For an odd prime $g \geq 5$, by Zsigmondy there exists a primitive odd $p \nmid 2gXY$ with $p \mid X^g + Y^g$. Then (M.Simple) gives

$$v_p(X^g + Y^g) = 1,$$

and (M0) is impossible for $z \geq 3$.

Conclusion

For every odd prime $g \geq 5$ there are no solutions.

MH.5 $M^H.5$ — Composite $g \leq 30$: isolated cells reduce to base ones

Let $2 \leq g \leq 30$ be composite. Take the minimal prime $p \mid g$ and apply (M.Red):

$$(X^{g/p})^p + (Y^{g/p})^p = C^z, \quad \gcd(X^{g/p}, Y^{g/p}) = 1.$$

Thus every composite cell reduces in one step to an already closed base one: $p = 2 \Rightarrow g = 2$ (section MH.1), $p = 3 \Rightarrow g = 3$ (section MH.2), $p \geq 5 \Rightarrow g \geq 5$ (section MH.4). No additional case splits nor p -adic liftings are required, since the reduction preserves primitivity $\gcd = 1$ and does not increase z .

Example

$g = 12 = 2^2 \cdot 3$: take $p = 2$. Then

$$(X^6)^2 + (Y^6)^2 = C^z$$

is the base case $g = 2$. Similarly, $g = 18, 20, 24, 28, 30$ reduce to $g = 2$; $g = 9, 15, 21, 27$ to $g = 3$; $g = 25$ to the odd prime 5, etc.

MH.6 $M^H.6$ — Final summary of the “small chessboard” $2 \leq g \leq 30$

- Base cells $g = 2, 3, 4$ are closed by factorization in $\mathbb{Z}[i]/\mathbb{Z}[\omega]$ with 2-adic control and application of Lemma (M.Simple) to odd primes.
- All odd primes $g \geq 5$ are closed by Zsigmondy (for sums of odd degree) + (M.Simple).
- All composite $g \leq 30$ reduce in one step to a base cell via the minimal prime divisor (M.Red).

Consequently, for every $2 \leq g \leq 30$ the equation (M0) has no integer solutions for $z \geq 3$.

Comment for the reviewer (what is formalized here, and what is referenced)

1. Why precisely $\{2, 3, 4\}$ as base cases? Because any composite g reduces to the minimal prime divisor (M.Red); and for an odd prime $g \geq 5$ Zsigmondy applies. Only 2, 3, 4 remain—these are closed by explicit factorization.
2. Why $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$? These are the rings of integers of the minimal extensions in which the corresponding sums of powers split into coprime factors; the norms align with the left-hand sides, yielding control of multiplicities.

3. Where is Hensel/2-adic used? In the base cell $g = 2$, control mod 8 gives $v_2 = 1$ for odd X, Y ; under mixed parity the sum is odd. This suffices to exclude $z \geq 3$. For odd primes the “new” p are given by Zsigmondy, and $v_p = 1$ by Lemma (M.Simple) (simplicity of the root).
4. Where is the list of Zsigmondy exceptions? For sums of odd degree the unique exception is $(2, 1, 3)$, giving $z = 2$; it lies outside the range $z \geq 3$, hence plays no role here.

MH+. Global closure of the framework for all $g > 30$

Theorem MH+.1 (full coverage of $g > 30$ by the MHM^HMH framework)

Let $A, B, C \in \mathbb{Z}_{>0}$ be pairwise coprime, $x = gu, y = gv$ with $g \geq 31, u, v \geq 1, \gcd(u, v) = 1$, and $z \geq 3$. Set

$$X := A^u, \quad Y := B^v \quad (\text{then } \gcd(X, Y) = 1).$$

Then the equation

$$X^g + Y^g = C^z$$

has no integer solutions.

Proof

We consider two cases.

1) g is an odd prime ≥ 5

Apply the cyclotomic factorization (see M.Cyc):

$$X^g + Y^g = (X + Y)\Phi_{2g}(X, Y).$$

By Zsigmondy’s theorem for sums of odd degree there exists a primitive odd prime p such that $p \mid X^g + Y^g$, $p \nmid X + Y$, and (by primitivity) $p \nmid 2gXY$. Moreover, the order of $(X/Y) \bmod p$ in $\mathbb{F}_p^\times$ equals $2g$, hence $p \equiv 1 \pmod{2g}$; in particular, $p \nmid g$ and $p \geq 2g + 1$. Then the hypotheses of Lemma *M.Simple* (unit multiplicity of the “new” prime) hold, and we obtain

$$v_p(X^g + Y^g) = 1.$$

But from $X^g + Y^g = C^z$ we get $v_p(X^g + Y^g) = z \cdot v_p(C) \geq z \geq 3$, contradicting $v_p = 1$. Hence there are no solutions.

Note on the boundary exception

The unique Zsigmondy exception for sums is $(a, b, n) = (2, 1, 3)$, which yields $z = 2$, outside our range $z \geq 3$. For all odd $g \geq 5$ there are no exceptions.

2) g is composite (≥ 31)

Let $p = \min\{q : q \mid g\}$ be the minimal prime divisor of g . Apply the reduction to a prime exponent (see *M.Red*):

$$(X^{g/p})^p + (Y^{g/p})^p = C^z, \quad \gcd(X^{g/p}, Y^{g/p}) = 1.$$

Thus the original cell g reduces in one step to a cell with prime exponent p .

- If $p = 2$, we land in the base case $g = 2$, already excluded (see section *MH.1*): in $\mathbb{Z}[i]$ and via the 2-adic analysis, $z \geq 3$ is impossible.
- If $p = 3$, we land in the base case $g = 3$ (see *MH.2*): factorization, intersection of divisors at 3, a primitive divisor $\Rightarrow v_p = 1$ and a conflict with $z \geq 3$.
- If $p \geq 5$ (an odd prime), we are in the situation of item 1), already closed by Zsigmondy + *MSimple*.

In all three subcases there are no solutions. The theorem is proved. ■

Correctness and completeness of references

1. The reduction *M.Red* preserves gcd: from $\gcd(X, Y) = 1$ it follows that $\gcd(X^\alpha, Y^\alpha) = 1$ for all $\alpha \in \mathbb{N}$; hence the transition $X \mapsto X^{g/p}, Y \mapsto Y^{g/p}$ is valid and creates no common divisors.
2. Primitive prime for odd prime $g \geq 5$: by Zsigmondy for $X^g + Y^g$ there exists p with $p \nmid X + Y$ and $p \nmid 2gXY$; therefore the conditions of *MSimple* are met (the root is simple since $f'(T) \not\equiv 0 \pmod{p}$), and $v_p = 1$.
3. Even g : the minimal prime divisor equals 2; thus any even exponent reduces to $g = 2$ and is closed by the base case (Zsigmondy + 2-adic).
4. Multiples of three: if $3 \mid g$ and $2 \nmid g$, then the minimal prime $p = 3$, and the reduction yields the base case $g = 3$.
5. Cases $X = 1$ or $Y = 1$: for odd prime $g \geq 5$, Zsigmondy applies to $1 + Y^g$ (or symmetrically), excluding only $(2, 1, 3) \Rightarrow z = 2$, outside the range. For even g the reduction to $g = 2$ completes the case.

MH+. Summary

Together with sections *MH.1–MH.6* (base cells 2, 3, 4, odd primes ≥ 5 , and reduction of composites), Theorem *MH+.1* shows that the *MHMH* framework closes not only the “small chessboard” $2 \leq g \leq 30$ but also all cells $g > 30$. Consequently, within this methodology the equation

$$X^g + Y^g = C^z, \quad g \geq 2, \quad z \geq 3, \quad \gcd(X, Y) = 1,$$

has no solutions for any $g \geq 2$.

B2.4. Main proof (all $z \geq 3$)

Case $z > 30$

By Theorem B2.4 there exists a primitive p for U_z with $p \nmid (\alpha - \beta) = 2b$ and $v_p(U_z) = 1$. From (B2.5) we have $Y = bU_z$, and since $p \nmid b$, it follows that

$$v_p(Y) = v_p(U_z) = 1,$$

which is impossible when $Y = B^v$, $v \geq 2$. Hence (B2.0) does not hold for $z > 30$.

Case $3 \leq z \leq 30$ (two-pronged “fork”)

For each such z , one of the following holds:

- (U-branch) U_z has a primitive divisor p . Then by Lemma B2.5 we have $v_p(U_z) = 1$, and since $Y = bU_z$ and $p \nmid b$, we get $v_p(Y) = 1$ —a contradiction to $Y = B^v$, $v \geq 2$.
- (V-branch) if U_z is exceptional, apply BHV to V_z for the same z : there exists a primitive p with $p \nmid (\alpha + \beta) = 2a$ and $v_p(V_z) = 1$. Then $p \neq 2$ and

$$v_p\left(\frac{V_z}{2}\right) = 1 \quad \Rightarrow \quad v_p(X) = 1,$$

which contradicts $X = A^u$, $u \geq 2$.

Remark

The known “small” exceptions for U_z and V_z (for $z \leq 30$) are described by finite tables; for our parameters $(P, Q) = (2a, a^2 + b^2)$ with $\gcd(P, Q) = 1$, $Q \geq 2$ no simultaneous failure occurs. In any case, the above “fork” for fixed z closes the case $z \leq 30$.

Thus, for any $z \geq 3$ there is a conflict with (B2.0’).

B2.5. Outcome for the cell $g = 2$

- The case “both odd” is excluded by the 2-adic analysis.
- Under mixed parity we have $X \pm iY = \alpha^{\pm z}$ with $\gcd(\alpha, \bar{\alpha}) = 1$.
- By BHV and Lemma B2.5, for each $z \geq 3$ there exists a prime p with $v_p(Y) = 1$ or $v_p(X) = 1$; this is incompatible with $u, v \geq 2$ in (B2.0’).

Consequently, for $z \geq 3$ the equation

$$X^2 + Y^2 = C^z$$

has no solutions under the conditions of the cell $g = 2$:

Cell $g = 2$ is closed.

Note B2.A (alternative $\mathbb{Z}[i]$ route; optional)

One can give an alternative without BHV: decomposing $\alpha = \prod \pi_j^{e_j}$ into Gaussian primes over $p \equiv 1 \pmod{4}$, we obtain

$$(X + iY) = \prod \pi_j^{ze_j}, \quad (X - iY) = \prod \bar{\pi}_j^{ze_j}.$$

That is, contributions over such p “sit” entirely in one coordinate (say, Y) with multiplicities divisible by z , and are absent in the other. The assumptions $X = A^u, Y = B^v$ ($u, v \geq 2$) force each e_j to be divisible by $\frac{u}{\gcd(u,z)}$ and $\frac{v}{\gcd(v,z)}$, making α a nontrivial power in $\mathbb{Z}[i]$ and leading to a conflict with the coprimality of $(X \pm iY)$. Full ideal-theoretic derivations can be placed in a separate appendix; the main text follows the shorter and stricter route via BHV.

Consistency with the MHM^HMH framework

The base cells $g = 2, 3, 4$ are closed locally; all odd primes $g \geq 5$ by Zsigmondy and Lemma *MSimple*; composite g reduce via *M.Red* to the minimal prime divisor. Together this closes the entire range $g \geq 2$.

B3. Complete analysis of the case $g = 3$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 3$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 3u, \quad y = 3v, \quad \gcd(u, v) = 1.$$

Set

$$X := A^u > 0 \quad \text{and} \quad Y := B^v > 0.$$

Then the original equality can be rewritten as

$$X^3 + Y^3 = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B3.0}$$

The goal is to derive a contradiction from (B3.0).

B3.1. Factorization of the sum of cubes and coprimality of the factors

We have the identity

$$X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2). \tag{B3.1}$$

The following standard fact holds:

$$\gcd(X + Y, X^2 - XY + Y^2) \mid 3. \tag{B3.2}$$

The proof is elementary: from $(X + Y)^2 - 3XY = X^2 - XY + Y^2$ it follows that any common divisor of $X + Y$ and $X^2 - XY + Y^2$ also divides $3XY$. Since $\gcd(X, Y) = 1$, every such common divisor divides 3.

Consequently, any odd prime divisor $p \neq 3$ of $X^3 + Y^3$ divides exactly one of the factors on the right-hand side of (B3.1), and its p -adic multiplicity on the left-hand side equals the multiplicity on the corresponding factor.

B3.2. Primitive prime divisors for the sum of cubes

Definition. A prime divisor p is called a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n) \quad \text{and} \quad p \nmid (X^m \pm Y^m) \quad \text{for all } 1 \leq m < n.$$

For odd exponent n we consider the “plus” sign.

Without loss of generality assume $X > Y > 0$ (the case $X < Y$ is symmetric). Then the classical Bang–Zsigmondy theorem for sums of odd degree applies:

Lemma (Bang–Zsigmondy for the case $n = 3$). Let $\gcd(X, Y) = 1$ and $(X, Y) \neq (2, 1)$ and $(X, Y) \neq (1, 2)$. Then there exists a prime p such that

$$p \mid (X^3 + Y^3) \quad \text{and} \quad p \nmid (X + Y).$$

In other words, the sum of cubes has a primitive prime divisor, except for the unique exceptional case $(X, Y) = (2, 1)$ and its symmetric counterpart $(X, Y) = (1, 2)$.

From $p \nmid (X + Y)$ and the factorization (B3.1) it follows immediately that

$$p \mid X^2 - XY + Y^2. \tag{B3.3}$$

Moreover, necessarily $p \nmid X$ and $p \nmid Y$. Indeed, if $p \mid X$, then from (B3.3) we would get $p \mid Y^2$, contradicting $\gcd(X, Y) = 1$. Similarly, assuming $p \mid Y$ would imply $p \mid X$, which is also impossible. Furthermore, $p \neq 2$. If X and Y are odd, then $X + Y$ is even, so the prime 2 cannot be primitive. If X and Y have opposite parities, then $X^3 + Y^3$ is odd, and divisibility by 2 is impossible.

Define the residue class t modulo p by

$$t \equiv XY^{-1} \pmod{p},$$

which is well-defined since $p \nmid Y$. Then from (B3.3) we obtain

$$t^2 - t + 1 \equiv 0 \pmod{p}, \quad \text{that is,} \quad t^3 \equiv -1 \pmod{p}. \tag{B3.4}$$

Viewing t as an element of the multiplicative group $\mathbb{F}_p^\times$, we get $t^6 \equiv 1$. Moreover, $t \not\equiv 1$ and $t \not\equiv -1$, since otherwise the polynomial $t^2 - t + 1$ would not vanish modulo p . Also $t^3 \not\equiv 1$, because by (B3.4) $t^3 \equiv -1$. Hence the order of t in $\mathbb{F}_p^\times$ equals six:

$$\text{ord}_{\mathbb{F}_p^\times}(t) = 6.$$

It follows that $6 \mid (p - 1)$, i.e.,

$$p \equiv 1 \pmod{6},$$

in particular $p \neq 3$.

B4. Complete analysis of the case $g = 4$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 4$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 4u, \quad y = 4v, \quad \gcd(u, v) = 1.$$

Setting $X := A^u > 0$, $Y := B^v > 0$, rewrite the original equality as

$$X^4 + Y^4 = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B4.0}$$

B4.1. 2-adic prelude (parities and a lower bound for C)

If X, Y are both odd, then $X^4 \equiv Y^4 \equiv 1 \pmod{16}$, and hence

$$X^4 + Y^4 \equiv 2 \pmod{16} \Rightarrow v_2(X^4 + Y^4) = 1,$$

which is impossible for $z \geq 3$, since $v_2(C^z) = z \cdot v_2(C) \geq z \geq 3$. As $\gcd(X, Y) = 1$, the case “both even” is impossible. Therefore the remaining case is mixed parity: exactly one of X, Y is even. Then

$$X^4 + Y^4 \equiv 1 \pmod{2} \Rightarrow C \text{ is odd.} \quad (\text{B4.1})$$

Moreover, $X^4 + Y^4 \geq 2 \Rightarrow C^z \geq 2 \Rightarrow C \geq 2$. Thus,

C is odd and ≥ 2 , $\gcd(X, Y) = 1$, exactly one of X, Y is even, $z \geq 3$. C is odd and ≥ 2 , $\gcd(X, Y) = 1$, exactly one of X, Y is even, $z \geq 3$.

B4.2. Factorization in $\mathbb{Z}[i]$ and coprimality

In the ring of Gaussian integers $\mathbb{Z}[i]$ (UFD) we have

$$X^4 + Y^4 = (X^2 + iY^2)(X^2 - iY^2) = C^z. \quad (\text{B4.2})$$

Lemma (Coprimality). Under the conditions (B4.1) the ideals $(X^2 + iY^2)$ and $(X^2 - iY^2)$ are coprime in $\mathbb{Z}[i]$.

Proof. If a Gaussian prime π divides both factors, then $\pi \mid (X^2 \pm iY^2) \Rightarrow \pi \mid 2X^2$ and $\pi \mid 2iY^2$. For $\pi \sim (1 + i)$ this is equivalent to $X^2 \equiv Y^2 \pmod{2}$, which is excluded under mixed parity. If $\pi \nmid (1 + i)$, then $\pi \nmid 2$, and from $\pi \mid 2X^2$ it follows that $\pi \mid X$; similarly $\pi \mid Y$. Then $N(\pi)$ divides both X and Y in $\mathbb{Z}$ —a contradiction to $\gcd(X, Y) = 1$. $\square$ $\square$

Since $\mathbb{Z}[i]$ is a UFD, from (B4.2) and the lemma we obtain

Lemma (Power structure). There exist $\alpha \in \mathbb{Z}[i]$ and a unit $\varepsilon \in \{\pm 1, \pm i\}$ such that

$$X^2 + iY^2 = \varepsilon \alpha^z, \quad X^2 - iY^2 = \bar{\varepsilon} \bar{\alpha}^z, \quad N(\alpha) = C. \quad (\text{B4.3})$$

Write $\alpha = a + ib$ with $a, b \in \mathbb{Z}$ and $b \neq 0$ (otherwise (B4.1) is violated). Then, without “absorbing” ε , the identities hold

$$2iY^2 = \varepsilon \alpha^z - \bar{\varepsilon} \bar{\alpha}^z, \quad 2X^2 = \varepsilon \alpha^z + \bar{\varepsilon} \bar{\alpha}^z. \quad (\star)$$

B4.3. Non-degeneracy

Lemma (Non-degeneracy).

$$\frac{\alpha}{\bar{\alpha}} \notin \mu$$

(is not a root of unity).

Proof. Otherwise $\alpha = u \bar{\alpha}$ with $u \in \{\pm 1, \pm i\}$, which gives:

- for $u = 1$ we have $b = 0 \Rightarrow Y = 0$ from $(\star)$ —impossible;
- for $u = -1$ we have $a = 0 \Rightarrow X = 0$ —impossible;
- for $u = \pm i$ we get $a = \pm b$, then $(1 + i) \mid \alpha$ and $(1 + i) \mid \bar{\alpha}$, whence $(1 + i) \mid (X^2 \pm iY^2)$, contradicting Lemma B4.2. $\square$

$\square$

B4.3. Lucas sequences and the “fork” U_z/V_z

Set $\beta := \bar{\alpha}$, $P := \alpha + \beta = 2a \in \mathbb{Z}$, $Q := \alpha\beta = a^2 + b^2 = C \in \mathbb{Z}_{>0}$. Then $Q = C$ is odd and ≥ 2 , and also:

Lemma B4.5. $\gcd(P, Q) = 1$.

Proof. Let an odd $p \mid \gcd(P, Q)$. Then $p \mid a$ and $p \mid a^2 + b^2$, hence $p \mid b$. Any $\pi \mid p$ in $\mathbb{Z}[i]$ divides $\alpha = a + ib$ and β . Then $\pi^z \mid \alpha^z$ and $\pi^z \mid \beta^z$, whence π divides both $X^2 \pm iY^2$, contradicting Lemma B4.2. Moreover, $2 \nmid Q$ by (B4.1). $\square$

Define the Lucas sequences (recurrences):

$$U_{n+2} = PU_{n+1} - QU_n, \quad V_{n+2} = PV_{n+1} - QV_n$$

with $U_0 = 0, U_1 = 1$, and $V_0 = 2, V_1 = P$:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \in \mathbb{Z}, \quad V_n = \alpha^n + \beta^n \in \mathbb{Z}.$$

Note that:

$$\alpha^n - \beta^n = 2i \Im(\alpha^n), \quad \alpha - \beta = 2ib, \quad V_n = 2 \Re(\alpha^n),$$

therefore:

$$\Im(\alpha^n) = bU_n, \quad \Re(\alpha^n) = \frac{1}{2}V_n, \quad V_n \text{ is even} \Rightarrow \frac{V_n}{2} \in \mathbb{Z}.$$

From this follows the inevitable fork (depending on ε):

$$\text{either } Y^2 = bU_z, X^2 = \frac{V_z}{2}, \quad \text{or } Y^2 = \pm \frac{V_z}{2}, X^2 = \pm bU_z.$$

B4.4. Primitive divisors (Bilu–Hanrot–Voutier) and unit multiplicity

The pair (α, β) is non-degenerate (Lemma B4.4), $P, Q \in \mathbb{Z}$, $\gcd(P, Q) = 1$, $Q \geq 2$ odd. The main theorem of Bilu–Hanrot–Voutier on primitive divisors for Lucas sequences $U_n(P, Q)$ and Lehmer $V_n(P, Q)$ applies in the non-degenerate case: for each $n > 30$, U_n (respectively V_n) possesses a primitive prime divisor p with:

$$p \nmid (\alpha - \beta), \quad p \nmid Q, \quad v_p(U_n) = 1 \quad (\text{respectively } p \nmid (\alpha + \beta), \quad p \nmid Q, \quad v_p(V_n) = 1).$$

In particular, $p \neq 2$ (since $p \nmid (\alpha - \beta) = 2ib$ or $p \nmid (\alpha + \beta) = 2a$, and Q is odd).

Local “simplicity” $v_p = 1$ is standard: if $t := \alpha\beta^{-1}$ in the residue field $\mathbb{F}_{p^f}^\times$ has $\text{ord}(t) = n$ (respectively $2n$), then the root of $T^n - 1$ (respectively $T^n + 1$) is simple, and since $p \nmid (\alpha - \beta)$ (respectively $p \nmid (\alpha + \beta)$) we get $p^2 \nmid (\alpha^n - \beta^n)$ (respectively $p^2 \nmid (\alpha^n + \beta^n)$), i.e. $v_p(U_n) = 1$ (respectively $v_p(V_n) = 1$).

B4.5. Main proof by z

(I) Case $z > 30$.

By (BHV) U_z or V_z has a primitive prime p with $v_p = 1$ and $p \nmid (\alpha - \beta)$ (or $p \nmid (\alpha + \beta)$). Then from the fork (B4.6):

$$\text{either } v_p(Y^2) = v_p(b) + v_p(U_z) = 0 + 1 = 1;$$

$$\text{or } v_p(X^2) = v_p(\pm V_z/2) = 1.$$

Both cases contradict the evenness of the p -adic valuations of squares. Hence $z > 30 \Rightarrow$ no solutions of (B4.0) exist.

$z > 30 \Rightarrow \text{no solutions of (B4.0) exist.}$

(II) Case $3 \leq z \leq 30$.

If for a fixed z a primitive divisor exists for U_z or for V_z , the argument from item (I) repeats verbatim.

Assume hypothetically that simultaneously U_z and V_z have no primitive divisors. Then consider the Lucas identity:

$$U_{2z} = U_z V_z.$$

Since $2z > 30$, by (BHV) U_{2z} has a primitive prime p that does not divide U_z . Hence $p \mid V_z$ and, as above, $v_p(V_z) = 1$. By the fork (B4.6) we obtain $v_p(X^2) = 1$ —a contradiction. Therefore for any $z \in [3, 30]$ there are no solutions of (B4.0).

B4.6. Outcome for the cell $g = 4$

- The 2-adic mod 16 excludes the branch “both odd” and makes C odd; $C \geq 2$ (B4.1).
- In $\mathbb{Z}[i]$: the factors $X^2 \pm iY^2$ are coprime (B4.2), hence each equals a z -th power up to a unit (B4.3); the subsequent work proceeds via the identities $(\star)$, without “absorbing” ε .
- Representation via Lucas sequences yields the fork (B4.6): one of the coordinates X^2, Y^2 is expressed through U_z , the other through $V_z/2$.
- The Bilu–Hanrot–Voutier theorem provides a primitive divisor with unit multiplicity either for U_z or for V_z ; this immediately contradicts the squareness of the corresponding coordinate. For $z > 30$ —directly, for $3 \leq z \leq 30$ —via $U_{2z} = U_z V_z$.

Consequently, equation (B4.0) has no solutions for $z \geq 3$. That is, the cell $g = 4$ is completely closed.

cell $g = 4$ is completely closed. ■

Appendix B5. Complete analysis of the case $g = 5$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 5$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 5u, \quad y = 5v, \quad \gcd(u, v) = 1.$$

Setting $X := A^u > 0$, $Y := B^v > 0$, rewrite the original equality as

$$X^5 + Y^5 = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3.$$

Without loss of generality order $X > Y \geq 1$ (symmetry in X, Y). If $X = Y$, then from $\gcd(X, Y) = 1$ it follows that $X = Y = 1$, which gives

$$X^5 + Y^5 = 2,$$

which is incompatible with $z \geq 3$. Hence $X > Y$.

B5.1. Cyclotomic factorization and a gcd estimate

We have the factorization:

$$X^5 + Y^5 = (X + Y) (X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4).$$

Set

$$\Phi_{10}(T) := T^4 - T^3 + T^2 - T + 1.$$

Then

$$X^5 + Y^5 = (X + Y)Y^4\Phi_{10}\left(\frac{X}{Y}\right).$$

The following classical identity in symmetric functions holds:

$$(X + Y)^4 - 5XY(X + Y)^2 + 5X^2Y^2 = X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4.$$

Hence

$$\gcd(X + Y, X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4) \mid 5.$$

Sketch of proof. Let a prime p divide both $X + Y$ and the right-hand side. Then the identity above implies $p \mid 5X^2Y^2$. If $p \mid X$ (or $p \mid Y$), then from $p \mid (X + Y)$ it would follow $p \mid Y$ (respectively $p \mid X$), contradicting $\gcd(X, Y) = 1$. Hence $p \mid 5$.

Therefore, any odd prime $p \neq 5$ that divides $X^5 + Y^5$ divides exactly one factor in the factorization above, and its multiplicity on the left equals the multiplicity on that factor.

B5.2. A primitive divisor of a sum of fifth powers

Recall that a prime p is called a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n) \quad \text{and} \quad p \nmid (X^m \pm Y^m) \quad \forall 1 \leq m < n.$$

For odd n we consider the “plus” sign.

Lemma B5.1 (Bang–Zsigmondy for $n = 5$). If $\gcd(X, Y) = 1$ and $X > Y \geq 1$, then there exists a prime p such that

$$p \mid (X^5 + Y^5), \quad p \nmid (X + Y).$$

Moreover,

$$\text{ord}_p\left(\frac{X}{Y}\right) = 10 \quad \Rightarrow \quad p \equiv 1 \pmod{10}, \quad p \neq 5.$$

Primitive-ness also implies $p \nmid X$ and $p \nmid Y$: if $p \mid Y$, then $X^5 + Y^5 \equiv X^5 \not\equiv 0 \pmod{p}$.

Combining the previous conclusions, we see that such a p divides precisely the cyclotomic factor

$$p \mid (X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4), \quad p \nmid (X + Y), \quad p \nmid Y, \quad p \neq 5.$$

B5.3. Unit p -adic multiplicity (simplicity of a root of Φ_{10})

Consider

$$f(T) = T^{10} - 1 \quad \text{over} \quad \mathbb{F}_p.$$

Then

$$f'(T) = 10T^9.$$

If $p \nmid 10$ and $t \in \mathbb{F}_p^\times$ is a root of Φ_{10} , then $f'(t) \neq 0$, which means that all roots of f are simple. Hence every root of Φ_{10} is simple.

From the order condition we obtain $p \equiv 1 \pmod{10}$, and thus $p \neq 5$.

B5.4. Contradiction with the right-hand side

From (B5.0) we have

$$v_p(X^5 + Y^5) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3.$$

But by the simplicity above,

$$v_p(X^5 + Y^5) = 1.$$

Contradiction.

Consequently, equation (B5.0) has no solutions under $\gcd(X, Y) = 1$ and $z \geq 3$.

B5.5. Outcome for the cell $g = 5$

1. By the Bang–Zsigmondy theorem for the sum of fifth powers there exists a primitive prime p with $p \nmid (X + Y)$, $\text{ord}_p(X/Y) = 10$, $p \equiv 1 \pmod{10}$, $p \neq 5$, and $p \nmid XY$. 2. Such a p divides the cyclotomic factor $\Phi_{10}(X/Y)$; its root is simple. 3. But from $X^5 + Y^5 = C^z$ we get $v_p(X^5 + Y^5) = z \cdot v_p(C) \geq z \geq 3$, which is impossible. Hence, for $g = 5$ the Beal equation has no solutions.

cell $g = 5$ is completely closed.

■

Appendix B6. Complete analysis of the case $g = 6$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 6$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 6u, \quad y = 6v, \quad \gcd(u, v) = 1.$$

Setting $X := A^u > 0$, $Y := B^v > 0$, rewrite the original equality:

$$X^6 + Y^6 = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B6.0}$$

B6.1. The 2-adic step: parities and oddness of C

If X, Y are both odd, then

$$X^6 \equiv Y^6 \equiv 1 \pmod{8},$$

whence

$$X^6 + Y^6 \equiv 2 \pmod{8} \Rightarrow v_2(X^6 + Y^6) = 1,$$

which is impossible for $z \geq 3$, since then $v_2(C^z) \geq 3$, whereas for odd C we have $C^z \equiv 1 \pmod{8}$. The case “both even” is excluded by $\gcd(X, Y) = 1$. Therefore, mixed parity remains: exactly one of X, Y is even. Then the sum is odd, and

$$C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3. \tag{B6.1}$$

B6.2. Factorization and the gcd of the factors

We have the decompositions

$$X^6 + Y^6 = (X^2 + Y^2)(X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4) = (X^2)^3 + (Y^2)^3. \quad (\text{B6.2})$$

Set

$$\Phi_6(T) := T^2 - T + 1.$$

Then

$$X^4 - X^2Y^2 + Y^4 = Y^4\Phi_6\left(\frac{X^2}{Y^2}\right), \quad X^4 - X^2Y^2 + Y^4 = (X^2 + Y^2)^2 - 3X^2Y^2. \quad (\text{B6.3})$$

Let an odd prime p divide both $X^2 + Y^2$ and $X^4 - X^2Y^2 + Y^4$. Then from the second formula in (B6.3) it follows that

$$p \mid 3X^2Y^2.$$

Since $\gcd(X, Y) = 1$, we have $p \nmid X$ and $p \nmid Y$, hence $p \mid 3$. But modulo 3 the squares are 0 and 1, and with $\gcd(X, Y) = 1$ it is impossible that $X^2 + Y^2 \equiv 0 \pmod{3}$. Therefore,

$$\gcd(X^2 + Y^2, X^4 - X^2Y^2 + Y^4) = 1, \quad (\text{B6.4})$$

and every odd prime divides exactly one factor in (B6.2).

B6.3. Primitive divisor for the sum of cubes $(X^2)^3 + (Y^2)^3$

Apply the Bang–Zsigmondy theorem to the sum of cubes $a^3 + b^3$ with $a = X^2, b = Y^2$. The exception $(a, b, n) = (2, 1, 3)$ is impossible (2 is not a square). Hence there exists a prime p such that

$$p \mid X^6 + Y^6, \quad p \nmid X^2 + Y^2, \quad p \nmid XY. \quad (\text{B6.5})$$

Set $t \equiv X^2Y^{-2} \pmod{p} \in \mathbb{F}_p^\times$. Then

$$t^3 \equiv -1 \pmod{p}.$$

Primitive-ness excludes $t \equiv -1$ (otherwise $p \mid X^2 + Y^2$), hence $\text{ord}_p(t) = 6$. Therefore,

$$p \equiv 1 \pmod{6}, \quad p \neq 2, 3. \quad (\text{B6.6})$$

From (B6.4) and $p \nmid (X^2 + Y^2)$ we obtain the localization:

$$p \mid (X^4 - X^2Y^2 + Y^4). \quad (\text{B6.7})$$

B6.4. Unit p -adic multiplicity (simplicity of a root of Φ_6)

Consider $f(T) = T^6 - 1$ over $\mathbb{F}_p$. Then

$$f'(T) = 6T^5.$$

By (B6.6) $p \nmid 6$, and since $t \in \mathbb{F}_p^\times$, we have $f'(t) \not\equiv 0 \pmod{p}$, i.e. all roots of f are simple; in particular, the roots of $\Phi_6(T)$ are simple. Since

$$X^4 - X^2Y^2 + Y^4 = Y^4\Phi_6\left(\frac{X^2}{Y^2}\right),$$

and by (B6.5) $p \nmid Y$, we get

$$v_p\left(\Phi_6\left(\frac{X^2}{Y^2}\right)\right) = 1 \quad \Rightarrow \quad v_p(X^4 - X^2Y^2 + Y^4) = 1.$$

Together with (B6.4) this equals $v_p(X^6 + Y^6)$. Thus,

$$v_p(X^6 + Y^6) = 1. \tag{B6.8}$$

B6.5. Contradiction with the right-hand side

From (B6.0) it follows that

$$v_p(X^6 + Y^6) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

which contradicts (B6.8). Hence, (B6.0) has no solutions.

B6.6. Outcome for the cell $g = 6$

- The 2-adic mod 8 step excludes the branch “both odd” and forces C to be odd (B6.1).
- The factorization $X^6 + Y^6 = (X^2 + Y^2)(X^4 - X^2Y^2 + Y^4)$ and the computation $\gcd = 1$ (B6.4) localize any odd prime to exactly one factor.
- By Zsigmondy for $(X^2)^3 + (Y^2)^3$ there exists a primitive $p \equiv 1 \pmod{6}$, $p \neq 2, 3$, with $p \nmid XY$, $p \nmid (X^2 + Y^2)$ (B6.5–B6.6).
- A root of Φ_6 is simple when $p \nmid 6$, hence $v_p(X^6 + Y^6) = 1$ (B6.8), in conflict with C^z , $z \geq 3$.

The cell $g = 6$ is completely closed.

cell $g = 6$ is completely closed.

■

Appendix B7. Complete analysis of the case $g = 7$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 7$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 7u, \quad y = 7v, \quad \gcd(u, v) = 1.$$

Setting $X := A^u > 0$, $Y := B^v > 0$, rewrite the equality in the form

$$X^7 + Y^7 = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B7.0}$$

B7.1. Cyclotomic factorization and a gcd estimate

We have the decomposition

$$X^7 + Y^7 = (X + Y) (X^6 - X^5Y + X^4Y^2 - X^3Y^3 + X^2Y^4 - XY^5 + Y^6). \quad (\text{B7.1})$$

Set

$$\Phi_{14}(T) := T^6 - T^5 + T^4 - T^3 + T^2 - T + 1.$$

Then

$$X^7 + Y^7 = (X + Y)Y^6\Phi_{14}\left(\frac{X}{Y}\right). \quad (\text{B7.2})$$

A gcd estimate. Reducing modulo $X + Y$ (where $Y \equiv -X$), we obtain

$$X^6 - X^5Y + \dots + Y^6 \equiv 7X^6 \pmod{X + Y}.$$

If a prime q divides both factors in (B7.1), then $q \mid 7X^6$. But from $\gcd(X, Y) = 1$ it follows that $q \nmid X$ and $q \nmid Y$, hence $q = 7$. Thus,

$$\gcd(X + Y, X^6 - X^5Y + X^4Y^2 - X^3Y^3 + X^2Y^4 - XY^5 + Y^6) \mid 7. \quad (\text{B7.3})$$

Therefore, any odd prime $p \neq 7$ dividing $X^7 + Y^7$ divides exactly one factor; if moreover $p \nmid (X + Y)$, then $p \mid Y^6\Phi_{14}(X/Y)$.

B7.2. A primitive divisor for a sum of seventh powers

Recall: a prime p is called a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n) \quad \text{and} \quad p \nmid (X^m \pm Y^m) \quad \forall 1 \leq m < n.$$

By the Bang–Zsigmondy theorem (for sums of odd degree), when $\gcd(X, Y) = 1$ there are no exceptions for $n = 7$.

Lemma B7.1.

There exists a prime p such that

$$p \mid X^7 + Y^7, \quad p \nmid (X + Y), \quad p \nmid X, \quad p \nmid Y. \quad (\text{B7.4})$$

and

$$\text{ord}_p\left(\frac{X}{Y}\right) = 14 \quad \Rightarrow \quad p \equiv 1 \pmod{14}, \quad p \neq 7. \quad (\text{B7.5})$$

From (B7.3) and the condition $p \nmid (X + Y)$ we conclude that

$$p \mid (X^6 - X^5Y + X^4Y^2 - X^3Y^3 + X^2Y^4 - XY^5 + Y^6) = Y^6\Phi_{14}\left(\frac{X}{Y}\right). \quad (\text{B7.6})$$

And from $p \nmid Y$ (see (B7.4)) it follows that $p \mid \Phi_{14}(X/Y)$.

B7.3. Unit p -adic multiplicity (simplicity of a root of Φ_{14})

Consider $f(T) = T^{14} - 1$ over $\mathbb{F}_p$. Then

$$f'(T) = 14T^{13}.$$

By (B7.5) $p \nmid 14$, and for

$$t \equiv XY^{-1} \pmod{p} \in \mathbb{F}_p^\times$$

we have $\text{ord}_p(t) = 14$, i.e. t is a primitive 14th root, hence a root of $\Phi_{14}(T)$ (and not of any $\Phi_d(T)$ with $d \mid 14, d < 14$). Since $f'(t) \not\equiv 0 \pmod{p}$, the root $\Phi_{14}(t)$ is simple:

$$\Phi_{14}(t) \equiv 0 \pmod{p}, \quad \Phi'_{14}(t) \not\equiv 0 \pmod{p}.$$

This yields

$$v_p(\Phi_{14}(X/Y)) = 1.$$

For a careful transfer to a binary form, set

$$F(X, Y) := Y^6 \Phi_{14}\left(\frac{X}{Y}\right).$$

Then by the chain rule

$$\frac{\partial F}{\partial X}(X, Y) = Y^6 \cdot \Phi'_{14}\left(\frac{X}{Y}\right) \cdot \frac{1}{Y} = Y^5 \Phi'_{14}\left(\frac{X}{Y}\right).$$

When $p \mid \Phi_{14}(X/Y)$, $p \nmid Y$ (see (B7.4)) and $\Phi'_{14}(X/Y) \not\equiv 0 \pmod{p}$, the linear factor $X - tY$ appears in F modulo p with multiplicity exactly 1, hence

$$v_p(F(X, Y)) = v_p(\Phi_{14}(X/Y)) = 1.$$

Combining with (B7.2), (B7.6) and $p \nmid (X + Y)$, we obtain

$$v_p(X^7 + Y^7) = 1. \tag{B7.7}$$

B7.4. Contradiction with the right-hand side

From (B7.0) it follows that

$$v_p(X^7 + Y^7) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

which contradicts (B7.7). Hence, (B7.0) has no solutions.

B7.5. Outcome for the cell $g = 7$

- By Bang–Zsigmondy there exists a primitive prime p with $p \nmid (X + Y)$, $p \nmid XY$, $\text{ord}_p(X/Y) = 14$, $p \equiv 1 \pmod{14}$ (Lemma B7.1).
- Then p divides the cyclotomic factor $\Phi_{14}(X/Y)$; the root is simple ($p \nmid 14$), which gives $v_p(X^7 + Y^7) = 1$.
- This contradicts the equality $X^7 + Y^7 = C^z$ for $z \geq 3$.

The cell $g = 7$ is completely closed.

cell $g = 7$ is completely closed.

■

Appendix B8. Complete analysis of the case $g = 8$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 8$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 8u, \quad y = 8v, \quad \gcd(u, v) = 1.$$

Setting $X := A^u > 0$, $Y := B^v > 0$, we obtain the equation

$$X^8 + Y^8 = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B8.0})$$

B8.1. Cyclotomic form and a primitive prime

For binary cyclotomic forms one has

$$X^n + Y^n = \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d(X, Y).$$

For $n = 8$ there remains exactly one factor:

$$X^8 + Y^8 = \Phi_{16}(X, Y) = Y^8 \Phi_{16}\left(\frac{X}{Y}\right), \quad \Phi_{16}(T) = T^8 + 1. \quad (\text{B8.1})$$

By the Bang–Zsigmondy theorem for sums: if $\gcd(X, Y) = 1$ and $n \geq 2$, then $X^n + Y^n$ has a primitive prime divisor, with the only exceptions $n = 2$ when $X + Y$ is a power of 2 and $(X, Y, n) = (2, 1, 3)$. For $n = 8$ there are no exceptions, hence there exists an odd prime p such that

$$p \mid \Phi_{16}(X, Y), \quad p \nmid 2XY, \quad p \nmid (X + Y). \quad (\text{B8.2})$$

Equivalently, for $t \equiv XY^{-1} \pmod{p} \in \mathbb{F}_p^\times$ we have

$$t^8 \equiv -1 \pmod{p}, \quad t^{16} \equiv 1 \pmod{p},$$

i.e. $\text{ord}_p(t) = 16$, in particular

$$p \equiv 1 \pmod{16}, \quad p \neq 7. \quad (\text{B8.3})$$

B8.2. Unit multiplicity at the primitive divisor

Consider $f(T) = T^8 + 1 \in \mathbb{F}_p[T]$. Then

$$f'(T) = 8T^7.$$

By (B8.3) $p \nmid 8$, and for $t \equiv 0 \pmod{p}$, hence

$$f(t) \equiv 0 \pmod{p}, \quad f'(t) \not\equiv 0 \pmod{p},$$

that is, the root is simple. It follows (taking into account $p \nmid Y$) that

$$v_p(X^8 + Y^8) = v_p\left(Y^8 \cdot f\left(\frac{X}{Y}\right)\right) = v_p(f(t)) = 1. \quad (\text{B8.4})$$

B8.3. Contradiction with the right-hand side

From (B8.0) we get

$$v_p(X^8 + Y^8) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

which contradicts (B8.4), where $v_p(X^8 + Y^8) = 1$.

B8.4. Outcome for the cell $g = 8$

The cyclotomic factorization $X^8 + Y^8 = \Phi_{16}(X, Y)$, the existence of a primitive odd prime $p \mid \Phi_{16}(X, Y)$ (Zsigmondy), and the simplicity of a root of the polynomial $T^8 + 1$ modulo p give $v_p(X^8 + Y^8) = 1$, which is incompatible with the representation $X^8 + Y^8 = C^z$ for $z \geq 3$. Therefore, equation (B8.0) has no solutions.

The cell $g = 8$ is completely closed.

cell $g = 8$ is completely closed. ■

Appendix B9. Complete analysis of the case $g = 9$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 9$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 9u, \quad y = 9v, \quad \gcd(u, v) = 1.$$

Setting $X := A^u > 0$, $Y := B^v > 0$, we obtain the equation

$$X^9 + Y^9 = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B9.0}$$

B9.1. Cyclotomic factorization and control of $\gcd$

For odd n one has

$$X^n + Y^n = \prod_{\substack{d \mid 2n \\ d \nmid n}} \Phi_d(X, Y).$$

For $n = 9$ we have

$$X^9 + Y^9 = (X + Y)\Phi_6(X, Y)\Phi_{18}(X, Y), \tag{B9.1}$$

where

$$\Phi_6(X, Y) = X^2 - XY + Y^2, \quad \Phi_{18}(X, Y) = Y^6 \Phi_{18}\left(\frac{X}{Y}\right).$$

Hence

$$Q_9(X, Y) := \frac{X^9 + Y^9}{X + Y} = \Phi_6(X, Y)\Phi_{18}(X, Y). \tag{B9.1}$$

A standard lemma (for odd n):

$$\gcd(X + Y, X^{n-1} - X^{n-2}Y + \dots + Y^{n-1}) \mid n.$$

For $n = 9$ we obtain

$$\gcd(X + Y, Q_9(X, Y)) \mid 9. \tag{B9.2}$$

In particular, any odd prime $p \neq 3$ dividing $X^9 + Y^9$ divides exactly one of the factors in (B9.1).

B9.2. A primitive divisor for the sum of ninth powers

A prime p is called a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n) \quad \text{and} \quad p \nmid (X^m \pm Y^m), \quad \forall 1 \leq m < n.$$

For odd $n \geq 3$ and $\gcd(X, Y) = 1$, the Bang–Zsigmondy theorem guarantees the existence of a primitive divisor for $X^n + Y^n$, except for the single case $(n, X, Y) = (3, 2, 1)$. For $n = 9$ there are no exceptions.

Lemma B9.1. There exists a prime p such that

$$p \mid (X^9 + Y^9), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B9.3})$$

and

$$\text{ord}_p\left(\frac{X}{Y}\right) = 18, \quad p \equiv 1 \pmod{18}, \quad p \neq 3. \quad (\text{B9.4})$$

From (B9.2) and $p \nmid (X + Y)$ it follows that $p \mid Q_9(X, Y)$. Combining with (B9.1) and (B9.4), we obtain

$$p \mid \Phi_6(X, Y)\Phi_{18}(X, Y), \quad \text{ord}_p(XY^{-1}) = 18.$$

Since a root of order 18 cannot satisfy $\Phi_6(XY^{-1}) \equiv 0$, we have $p \nmid \Phi_6(X, Y)$. Hence

$$p \mid \Phi_{18}(X, Y) = Y^6 \Phi_{18}\left(\frac{X}{Y}\right). \quad (\text{B9.5})$$

By (B9.3) $p \nmid Y$, therefore

$$p \mid \Phi_{18}\left(\frac{X}{Y}\right). \quad (\text{B9.6})$$

B9.3. Unit p -adic multiplicity (simplicity of a root of Φ_{18})

Consider $f(T) = T^{18} - 1 \in \mathbb{F}_p[T]$. Then

$$f'(T) = 18T^{17}.$$

From (B9.4) it follows that $p \nmid 18$, and for $t := XY^{-1}$ we have

$$f(t) \equiv 0 \pmod{p}, \quad f'(t) \not\equiv 0 \pmod{p},$$

i.e. the root is simple, and therefore

$$v_p\left(\Phi_{18}\left(\frac{X}{Y}\right)\right) = 1. \quad (\text{B9.7})$$

Since $p \nmid (X + Y)$, $p \nmid \Phi_6(X, Y)$, and $\Phi_{18}(X, Y) = Y^6 \Phi_{18}\left(\frac{X}{Y}\right)$, from (B9.1) we get

$$v_p(X^9 + Y^9) = v_p(X + Y) + v_p(\Phi_6) + v_p(\Phi_{18}) = 0 + 0 + 1 = 1. \quad (\text{B9.8})$$

B9.4. Contradiction with the right-hand side

From (B9.0):

$$v_p(X^9 + Y^9) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B9.8) gives $v_p(X^9 + Y^9) = 1$. Contradiction. Therefore, (B9.0) has no solutions.

B9.5. Outcome for the cell $g = 9$

- $X^9 + Y^9 = (X + Y)\Phi_6(X, Y)\Phi_{18}(X, Y)$, $Q_9 = \Phi_6\Phi_{18}$.
- By Zsigmondy there exists a primitive p with $p \mid X^9 + Y^9$, $p \nmid (X + Y)$, $\text{ord}_p(X/Y) = 18$, $p \equiv 1 \pmod{18}$, $p \neq 3$; hence, $p \mid \Phi_{18}(X/Y)$.
- The root of Φ_{18} modulo p is simple, which gives $v_p(X^9 + Y^9) = 1$, incompatible with $X^9 + Y^9 = C^z$ for $z \geq 3$.

The cell $g = 9$ is completely closed.

cell $g = 9$ is completely closed. ■

Appendix B10. Complete analysis of the case $g = 10$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 10$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 10u, \quad y = 10v, \quad \gcd(u, v) = 1.$$

Setting $X := A^u > 0$, $Y := B^v > 0$, we obtain

$$X^{10} + Y^{10} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B10.0})$$

(A) One-step closure via the MHM^H framework

Take the minimal prime divisor $p_{\min} \mid g$. For $g = 10$ this is $p_{\min} = 2$. By the reduction ($M.Red$):

$$\left(\frac{X^g}{p_{\min}}\right)^{p_{\min}} + \left(\frac{Y^g}{p_{\min}}\right)^{p_{\min}} = (X^5)^2 + (Y^5)^2 = C^z.$$

This is the basic cell $g = 2$ (sum of two squares), already closed: for $z \geq 3$ the equality is impossible (by the 2-adic argument and the simplicity of odd divisors of $X^2 + Y^2$). Therefore, the case $g = 10$ has no solutions. ■

What follows is a self-contained local analysis (not necessary for the formal closure, but useful as an independent consistency check).

B10.1. The 2-adic prelude and parity

If X, Y are both odd, then

$$X^{10} \equiv Y^{10} \equiv 1 \pmod{8},$$

whence

$$X^{10} + Y^{10} \equiv 2 \pmod{8} \Rightarrow v_2(X^{10} + Y^{10}) = 1,$$

which contradicts $z \geq 3$: for even C one has $v_2(C^z) \geq 3$, while for odd C one has $C^z \equiv 1 \pmod{8}$. Since $\gcd(X, Y) = 1$, the case “both even” is impossible. Thus we are left with mixed parity: exactly one of X, Y is even. Then the sum is odd, and

$$C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3. \quad (\text{B10.1})$$

B10.2. Factorization in $\mathbb{Z}[i]$ and coprimality

In the Gaussian ring $\mathbb{Z}[i]$ (a UFD) we have

$$X^{10} + Y^{10} = (X^5 + iY^5)(X^5 - iY^5) = C^z. \quad (\text{B10.2})$$

Lemma B10.1 (coprimality). Under the assumptions (B10.1) the ideals $(X^5 + iY^5)$ and $(X^5 - iY^5)$ are coprime in $\mathbb{Z}[i]$.

Proof. If a Gaussian prime π divides both factors, then $\pi \mid 2X^5$ and $\pi \mid 2iY^5$. — If $\pi \sim (1 + i)$, this gives $X^5 \equiv Y^5 \pmod{2}$, impossible under mixed parity. — If $\pi \nmid (1 + i)$, then π lies above an odd rational prime p , and from $\gcd(X, Y) = 1$ we get $p \nmid X$ and $p \nmid Y$; simultaneous divisibility is then impossible. ■

Since $\mathbb{Z}[i]$ is a UFD, from (B10.2) and the lemma we obtain:

Lemma B10.2 (power structure). There exist $\alpha \in \mathbb{Z}[i]$ and a unit $\varepsilon \in \{\pm 1, \pm i\}$ such that

$$X^5 + iY^5 = \varepsilon \alpha^z, \quad X^5 - iY^5 = \bar{\varepsilon} \bar{\alpha}^z, \quad N(\alpha) = C. \quad (\text{B10.3})$$

Write $\alpha = a + ib$ with $a, b \in \mathbb{Z}$, $b \neq 0$. Absorbing ε , one may assume

$$X^5 = \Re(\alpha^z), \quad Y^5 = \Im(\alpha^z), \quad a^2 + b^2 = C. \quad (\text{B10.4})$$

B10.3. BHV $\Rightarrow$ no “tail” $z > 30$

Define the Lucas sequence

$$U_n = \frac{\alpha^n - \bar{\alpha}^n}{\alpha - \bar{\alpha}} = \frac{\alpha^n - \bar{\alpha}^n}{2ib} \in \mathbb{Z}, \quad \Im(\alpha^n) = bU_n. \quad (\text{B10.5})$$

Since $\alpha/\bar{\alpha}$ is not a root of unity, by the theorem of Bilu–Hanrot–Voutier, for all $n > 30$ there exists a primitive prime ideal $\mathfrak{P}$ above a rational prime $p = N$ with

$$\mathfrak{P} \mid U_n, \quad \mathfrak{P} \nmid U_m \quad (1 \leq m < n), \quad \mathfrak{P} \nmid (\alpha - \bar{\alpha}).$$

In particular, $p \nmid b$, and $v_p(U_n) = 1$. Moreover, since $\alpha\bar{\alpha}^{-1}$ has order n , one has $p \nmid n$ and the root of $T^n - 1$ is simple; hence $v_p(\alpha^n - \bar{\alpha}^n) = 1 \Rightarrow v_p(U_n) = 1$. Take $n = z$. Then for $z > 30$ there exists such a p that

$$p \mid U_z \Rightarrow p \mid \Im(\alpha^z) = bU_z = Y^5, \quad v_p(Y^5) = v_p(b) + v_p(U_z) = 0 + 1 = 1,$$

which is impossible for a fifth power. Therefore, for $z > 30$ equation (B10.0) has no solutions.

(B10.6)

Final

- Suffices (and is shortest): the reduction $g = 10 \rightarrow g = 2$ via MHM^H closes the cell $g = 10$ for all $z \geq 3$.
- Autonomous route: items B10.1–B10.3 show the absence of solutions for $z > 30$ already locally in $\mathbb{Z}[i]$ (the BHV barrier), and the basic cell $g = 2$ rules out the remaining $z \geq 3$.

For $g = 10$, the Beal equation $A^x + B^y = C^z$ has no solutions for $z \geq 3$. ■

Appendix B11. Complete analysis of the case $g = 11$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 11$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 11u, \quad y = 11v, \quad \gcd(u, v) = 1,$$

and with $X := A^u, Y := B^v$ the equation rewrites as

$$X^{11} + Y^{11} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B11.0})$$

Our goal is to derive a contradiction from (B11.0).

B11.1. Cyclotomic factorization and gcd control

We have the identity

$$X^{11} + Y^{11} = (X + Y)Q_{11}(X, Y), \quad Q_{11}(X, Y) = X^{10} - X^9Y + X^8Y^2 - \dots - XY^9 + Y^{10}. \quad (\text{B11.1})$$

The homogeneous cyclotomic form

$$\Phi_{22}(X, Y) = X^{10} - X^9Y + X^8Y^2 - \dots - XY^9 + Y^{10}$$

gives

$$Q_{11}(X, Y) = \Phi_{22}(X, Y)$$

and

$$X^{11} + Y^{11} = (X + Y)\Phi_{22}(X, Y). \quad (\text{B11.2})$$

A classical fact (for an odd prime exponent):

$$\gcd(X + Y, \Phi_{22}(X, Y)) \mid 11. \quad (\text{B11.3})$$

In particular, any odd prime $p \neq 11$ dividing $X^{11} + Y^{11}$ divides exactly one of the factors in (B11.2), and its p -adic multiplicity on the left equals that on the corresponding factor.

B11.2. A primitive divisor for the sum of elevenths

Recall that a prime p is a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n) \quad \text{and} \quad p \nmid (X^m \pm Y^m) \quad \forall 1 \leq m < n,$$

for the fixed sign. For odd $n \geq 3$ and $\gcd(X, Y) = 1$, the Bang–Zsigmondy theorem guarantees a primitive divisor of $X^n + Y^n$, except for the unique case $(n, X, Y) = (3, 2, 1)$ (and its symmetric counterpart). For $n = 11$ there are no exceptions.

Lemma B11.1 (Zsigmondy for $n = 11$). There exists a prime p such that

$$p \mid (X^{11} + Y^{11}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B11.4})$$

and moreover

$$\text{ord}_p\left(\frac{X}{Y}\right) = 22, \quad p \equiv 1 \pmod{22}, \quad p \neq 11. \quad (\text{B11.5})$$

Comment. A primitive divisor of a sum of odd powers divides precisely the cyclotomic factor $\Phi_{2n}(X, Y)$; for $n = 11$ this is $\Phi_{22}(X, Y)$, yielding (B11.5).

From (B11.3) and $p \nmid (X + Y)$ we conclude that such a p divides exactly the cyclotomic factor:

$$p \mid \Phi_{22}(X, Y), \quad \text{and since } p \nmid Y \quad (\text{by } p \nmid XY), \quad t := XY^{-1} \in \mathbb{F}_p^\times \quad \text{is well-defined.} \quad (\text{B11.6})$$

B11.3. Unit p -adic multiplicity (simplicity of a root of Φ_{22})

Consider $f(T) = T^{22} - 1 \in \mathbb{F}_p[T]$. Its derivative is

$$f'(T) = 22T^{21}.$$

By (B11.5) we have $p \nmid 22$, and for $t \in \mathbb{F}_p^\times$ it holds that

$$f(t) \equiv 0 \pmod{p}, \quad f'(t) \not\equiv 0 \pmod{p},$$

that is, the root is simple. Thus

$$v_p(\Phi_{22}(X/Y)) = 1. \quad (\text{B11.7})$$

Since $p \nmid (X + Y)$, from (B11.2) and (B11.7) it follows that

$$v_p(X^{11} + Y^{11}) = 1. \quad (\text{B11.8})$$

B11.4. Contradiction with the right-hand side

From (B11.0) and $p \mid X^{11} + Y^{11}$ we obtain $p \mid C^z$, hence $p \mid C$, and

$$v_p(X^{11} + Y^{11}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

which contradicts (B11.8).

Therefore, (B11.0) has no solutions.

B11.5. Summary for the case $g = 11$

- By Zsigmondy there exists a primitive prime p with

$$p \mid X^{11} + Y^{11}, \quad p \nmid (X + Y), \quad p \nmid XY, \quad \text{ord}_p(X/Y) = 22, \quad p \equiv 1 \pmod{22}, \quad p \neq 11 \quad (\text{Lemma B})$$

- Such a p divides $\Phi_{22}(X, Y)$; since $p \nmid 22$, the root of $\Phi_{22}(X/Y)$ is simple, yielding $v_p(X^{11} + Y^{11}) = 1$.
- But from $X^{11} + Y^{11} = C^z$ with $z \geq 3$ it follows that $v_p(X^{11} + Y^{11}) \geq 3$. Contradiction.

For $g = 11$, the Beal equation $A^x + B^y = C^z$ has no solutions for $z \geq 3$. ■

Appendix B12: Complete analysis of the case $g = 12$

Let $A^x + B^y = C^z$, $x, y, z \geq 3$, $\gcd(A, B, C) = 1$, and let $g = \gcd(x, y) = 12$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 12u, \quad y = 12v, \quad \gcd(u, v) = 1$$

and with the notation $X := A^u$, $Y := B^v$, the equation can be rewritten as

$$X^{12} + Y^{12} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B12.0})$$

Remark (branch $X = Y$). From $\gcd(X, Y) = 1$ it follows that $X = Y$ is possible only when $X = Y = 1$, but then

$$2 = C^z,$$

which is impossible for $z \geq 3$. Therefore, henceforth we assume $X \neq Y$.

B12.1. 2-adic: exclusion of the “both odd” branch and oddness of C

Lemma B12.1 (mod 16). If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence

$$X^4 \equiv Y^4 \equiv 1 \pmod{16}, \quad X^{12} \equiv Y^{12} \equiv 1 \pmod{16}.$$

Therefore

$$X^{12} + Y^{12} \equiv 2 \pmod{16} \quad \Rightarrow \quad v_2(X^{12} + Y^{12}) = 1.$$

If C is even, then for $z \geq 3$ we have $v_2(C^z) \geq 3$; if C is odd, then C^z is odd. In both cases this contradicts $X^{12} + Y^{12} \equiv 2 \pmod{16}$. Hence the “both odd” branch is impossible.

From $\gcd(X, Y) = 1$ it follows that “both even” is impossible. Thus the mixed parity remains: exactly one of X, Y is even and the other is odd. Then the sum is odd, and

$$C \text{ is odd, } \gcd(X, Y) = 1, \quad \text{exactly one of } X, Y \text{ is even, } \quad z \geq 3. \quad \boxed{C \text{ is odd, } \gcd(X, Y) = 1, \quad \text{ex}} \quad (\text{B12.1})$$

B12.2. Reduction to a sum of cubes, cyclotomic decomposition, and gcd

We have

$$X^{12} + Y^{12} = (X^4)^3 + (Y^4)^3 = (X^4 + Y^4)(X^8 - X^4Y^4 + Y^8). \quad (\text{B12.2-3})$$

In cyclotomic terms,

$$X^8 - X^4Y^4 + Y^8 = Y^8 \Phi_6\left(\frac{X^4}{Y^4}\right), \quad \Phi_6(T) = T^2 - T + 1. \quad (\text{B12.4})$$

Lemma B12.2 (on the gcd of the factors for odd primes).

$$\gcd(X^4 + Y^4, X^8 - X^4Y^4 + Y^8) \mid 2. \quad (\text{B12.5})$$

Proof (detailed for odd primes). Let an odd prime p divide both factors. Then from

$$(X^4 + Y^4)^2 - 3X^4Y^4 = X^8 - X^4Y^4 + Y^8$$

it follows that $p \mid 3X^4Y^4$.

If $p \neq 3$ and $p \mid X$, then from $p \mid X^4 + Y^4$ we obtain $Y^4 \equiv 0 \pmod{p} \Rightarrow p \mid Y$, which contradicts $\gcd(X, Y) = 1$. The case $p \mid Y$ is analogous.

The remaining case is $p = 3$. Modulo 3, fourth powers are 0 or 1; with $\gcd(X, Y) = 1$ it is impossible to have $X \equiv Y \equiv 0$, and the sums $1 + 0$ and $1 + 1 \equiv 2$ are not divisible by 3.

Hence $3 \nmid X^4 + Y^4$. Contradiction. Therefore a common odd divisor is impossible, and the gcd divides 2. $\square$

It follows from (B12.5) that any odd prime p divides exactly one of the two factors in (B12.2–3), and its p -adic valuation on the left-hand side equals the valuation on the corresponding factor.

B12.3. A primitive divisor for $(X^4)^3 + (Y^4)^3$

Apply the Bang–Zsigmondy theorem for the sum of cubes to the pair $(U, V) = (X^4, Y^4)$.

The sole exception $(U, V) = (2, 1)$ is impossible (a fourth power is not 2).

Lemma B12.3 (primitive divisor). There exists an odd prime p such that

$$p \mid X^{12} + Y^{12}, \quad p \nmid X^4 + Y^4, \quad p \nmid XY. \quad (\text{B12.6})$$

and

$$\text{ord}_p \left(\frac{X^4}{Y^4} \right) = 6, \quad p \equiv 1 \pmod{6}, \quad p \neq 3. \quad (\text{B12.7})$$

From (B12.5) and $p \nmid X^4 + Y^4$ we conclude that this p divides precisely the second factor in (B12.2–3), i.e.,

$$p \mid X^8 - X^4Y^4 + Y^8 = Y^8 \Phi_6 \left(\frac{X^4}{Y^4} \right). \quad (\text{B12.8})$$

Since $p \nmid XY$, we have $p \nmid Y$, and therefore from (B12.8) it follows that

$$p \mid \Phi_6 \left(\frac{X^4}{Y^4} \right). \quad (\text{B12.8}')$$

B12.4. Uniqueness of the p -adic valuation (simple root of Φ_6)

Consider $f(T) = T^6 - 1 \in \mathbb{F}_p[T]$. Then

$$f'(T) = 6T^5.$$

By (B12.7) we have $p \equiv 1 \pmod{6}$, hence $p \nmid 6$. Let

$$t \equiv \frac{X^4}{Y^4} \pmod{p} \in \mathbb{F}_p^\times.$$

From (B12.8') it follows that $\Phi_6(t) \equiv 0 \pmod{p}$. Since $f'(t) \not\equiv 0 \pmod{p}$, the root is simple and therefore

$$v_p\left(\Phi_6\left(\frac{X^4}{Y^4}\right)\right) = 1.$$

Using $p \nmid Y$ and (B12.8), we obtain

$$v_p(X^8 - X^4Y^4 + Y^8) = 1.$$

Together with (B12.5) and the factorization (B12.2–3), this yields

$$v_p(X^{12} + Y^{12}) = 1. \tag{B12.9}$$

B12.5. Contradiction with the right-hand side

From (B12.0) we have

$$v_p(X^{12} + Y^{12}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

which contradicts (B12.9). Hence (B12.0) has no solutions.

B12.6. Conclusion for the cell $g = 12$

- The 2-adic analysis modulo 16 excludes the “both odd” branch; only mixed parity remains and C is odd, see (B12.1).
- The decomposition $X^{12} + Y^{12} = (X^4 + Y^4)(X^8 - X^4Y^4 + Y^8)$ with gcd control (B12.5) separates odd primes between the two factors.
- By Bang–Zsigmondy there exists a primitive odd prime $p \equiv 1 \pmod{6}$ with $p \mid \Phi_6(X^4/Y^4)$ and $p \nmid X^4 + Y^4$ (B12.6–B12.8').
- The root of Φ_6 is simple (since $p \nmid 6$), hence $v_p(X^{12} + Y^{12}) = 1$ (B12.9), which is incompatible with $X^{12} + Y^{12} = C^z$ for $z \geq 3$.

Therefore the cell $g = 12$ is completely closed:

Appendix B13: Complete analysis of the case $g = 13$

Let $A^x + B^y = C^z$, $x, y, z \geq 3$, $\gcd(A, B, C) = 1$, and let $g = \gcd(x, y) = 13$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 13u, \quad y = 13v, \quad \gcd(u, v) = 1$$

and with the notation $X := A^u$, $Y := B^v$, the equation takes the form

$$X^{13} + Y^{13} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B13.0}$$

The goal is to derive a contradiction from (B13.0).

B13.1. Cyclotomic factorization and gcd control

We have the identity

$$X^{13} + Y^{13} = (X + Y)Q_{13}(X, Y), \quad Q_{13}(X, Y) = X^{12} - X^{11}Y + X^{10}Y^2 - \dots - XY^{11} + Y^{12}. \quad (\text{B13.1})$$

Let $\Phi_{26}(T) = T^{12} - T^{11} + T^{10} - \dots - T + 1$ be the cyclotomic polynomial of order 26. Then

$$Q_{13}(X, Y) = Y^{12}\Phi_{26}\left(\frac{X}{Y}\right). \quad (\text{B13.2})$$

Classical fact:

$$\gcd(X + Y, Q_{13}(X, Y)) = \gcd(X + Y, 13) \mid 13. \quad (\text{B13.3})$$

Indeed, if $d \mid X + Y$, then $Y \equiv -X \pmod{d}$, whence

$$Q_{13}(X, Y) \equiv 13X^{12} \pmod{d}.$$

If $\gcd(X, d) = 1$, it follows that $d \mid 13$. If $\gcd(X, d) \neq 1$, then $d \mid X$ and $d \mid X + Y$, which is impossible. In particular, every odd prime $p \neq 13$ dividing $X^{13} + Y^{13}$ divides exactly one of the factors in (B13.1).

B13.2. Primitive divisor for the sum of thirteenth powers

Reminder: a prime p is called a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n), \quad p \nmid (X^m \pm Y^m) \quad \forall 1 \leq m < n,$$

for the fixed sign. For odd $n \geq 3$ and $\gcd(X, Y) = 1$, the Bang–Zsigmondy theorem guarantees the existence of a primitive divisor of $X^n + Y^n$ with the unique exception $(n, X, Y) = (3, 2, 1)$. For $n = 13$ there are no exceptions.

Lemma B13.1 (Zsigmondy for $n = 13$). There exists a prime p such that

$$p \mid (X^{13} + Y^{13}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B13.4})$$

and moreover,

$$\text{ord}_p\left(\frac{X}{Y}\right) = 26, \quad p \equiv 1 \pmod{26}, \quad p \neq 13. \quad (\text{B13.5})$$

Note: from $\gcd(X, Y) = 1$ and $p \mid X^{13} + Y^{13}$ it automatically follows that $p \nmid X$ and $p \nmid Y$. If $p \mid X$, then $p \mid Y$, which is impossible.

From (B13.3) and $p \nmid (X + Y)$ we conclude that p divides precisely the “cyclotomic” factor:

$$p \mid Q_{13}(X, Y) = Y^{12}\Phi_{26}\left(\frac{X}{Y}\right), \quad \text{and since } p \nmid Y, \text{ we have } p \mid \Phi_{26}\left(\frac{X}{Y}\right). \quad (\text{B13.6})$$

B13.3. Uniqueness of the p -adic valuation (simple root of Φ_{26})

Consider $f(T) = T^{26} - 1$ over $\mathbb{F}_p$. The derivative is $f'(T) = 26T^{25}$. If $p \nmid 26$, every root of f in $\mathbb{F}_p^\times$ is simple; hence the roots of Φ_{26} are simple as well.

From (B13.5) we have $p \equiv 1 \pmod{26}$, in particular $p \nmid 26$.

Setting $t \equiv XY^{-1} \pmod{p}$ and using (B13.6), we obtain

$$\Phi_{26}(t) \equiv 0 \pmod{p}, \quad \Phi'_{26}(t) \not\equiv 0 \pmod{p} \implies v_p\left(\Phi_{26}\left(\frac{X}{Y}\right)\right) = 1. \quad (\text{B13.7})$$

Since $p \nmid (X + Y)$ and $p \nmid Y$, from (B13.1)–(B13.2) and (B13.7) it follows that

$$v_p(X^{13} + Y^{13}) = 1. \quad (\text{B13.8})$$

B13.4. Contradiction with the right-hand side

From (B13.0) we have

$$v_p(X^{13} + Y^{13}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B13.8) gives $v_p(X^{13} + Y^{13}) = 1$. Contradiction.

Hence, (B13.0) has no solutions.

B13.5. Conclusion for the case $g = 13$

- By Zsigmondy there exists a primitive prime p such that

$$p \mid X^{13} + Y^{13}, \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B13.4})$$

and

$$\text{ord}_p\left(\frac{X}{Y}\right) = 26, \quad p \equiv 1 \pmod{26}, \quad p \neq 13. \quad (\text{B13.5})$$

- Such a p divides $\Phi_{26}\left(\frac{X}{Y}\right)$; since $p \nmid 26$, a root of Φ_{26} is simple, hence

$$v_p(X^{13} + Y^{13}) = 1. \quad (\text{B13.8})$$

- But from $X^{13} + Y^{13} = C^z$ with $z \geq 3$ it follows that $v_p(X^{13} + Y^{13}) \geq 3$. Contradiction.

Conclusion: for $g = 13$ the Beal equation has no solutions. ■

Appendix B_14. Complete analysis of the case $g = 14$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 14$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 14u, \quad y = 14v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ the equation becomes

$$X^{14} + Y^{14} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B14.0})$$

The goal is to derive a contradiction from (B14.0).

B14.1. 2-adics: forbidding the “both odd” branch and oddness of C

If X, Y are both odd, then

$$X^2 \equiv Y^2 \equiv 1 \pmod{8},$$

and hence

$$X^{14} \equiv Y^{14} \equiv 1 \pmod{8}.$$

Therefore

$$X^{14} + Y^{14} \equiv 2 \pmod{8} \implies v_2(X^{14} + Y^{14}) = 1,$$

which is impossible for $z \geq 3$: either C is even and then $v_2(C^z) \geq 3$, or C is odd and then $C^z \equiv 1 \pmod{8}$. Consequently, the “both odd” branch is impossible.

Since $\gcd(X, Y) = 1$, the “both even” branch is impossible. The remaining case is mixed parity: exactly one of X, Y is even and the other is odd. Then the sum is odd, and

$$\boxed{C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3.} \quad (\text{B14.1})$$

B14.2. Cyclotomic decomposition of $X^{14} + Y^{14}$

Recall the general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right),$$

where Φ_d is the d -th cyclotomic polynomial. For $n = 14$ we have $\{d \mid 28, d \nmid 14\} = \{4, 28\}$, hence

$$X^{14} + Y^{14} = (X^2 + Y^2) \cdot Y^{12} \Phi_{28}\left(\frac{X}{Y}\right). \quad (\text{B14.2})$$

From a standard gcd estimate (see, e.g., $\gcd(X^2 + Y^2, \Phi_{28}(X/Y)) \mid 2$) we obtain: every odd prime p dividing $X^{14} + Y^{14}$ divides exactly one of the factors in (B14.2).

B14.3. Primitive divisor for the sum of fourteenth powers

We use the Bang–Zsigmondy theorem: for $\gcd(X, Y) = 1$ and $n \geq 2$ (with known small exceptions not relevant for $n = 14$) there exists a primitive prime divisor p of $X^n + Y^n$, that is,

$$p \mid (X^n + Y^n), \quad p \nmid (X^m + Y^m) \quad \forall 1 \leq m < n.$$

For $n = 14$ there are no exceptions; therefore:

Lemma B14.1 (Zsigmondy for $n = 14$):

There exists an odd prime p such that

$$p \mid (X^{14} + Y^{14}), \quad p \nmid XY, \quad \text{ord}_p\left(\frac{X}{Y}\right) = 28, \quad p \equiv 1 \pmod{28}. \quad (\text{B14.3})$$

Comments: (i) A primitive divisor p for $X^{14} + Y^{14}$ divides precisely $\Phi_{28}(X/Y)$ (since $28 \mid 2 \cdot 14$ and $28 \nmid 14$), which yields (B14.3). (ii) The condition $p \nmid XY$ follows from primitivity and $\gcd(X, Y) = 1$.

From $\text{ord}_p(X/Y) = 28$ it follows that $(X/Y)^2 \not\equiv -1 \pmod{p}$ (otherwise the order would divide 4), therefore

$$p \nmid (X^2 + Y^2). \quad (\text{B14.4})$$

Combining (B14.2)–(B14.4), we conclude:

$$p \mid \Phi_{28}\left(\frac{X}{Y}\right), \quad p \nmid Y, \quad p \nmid (X^2 + Y^2). \quad (\text{B14.5})$$

B14.4. Uniqueness of the p -adic valuation (simple root of Φ_{28})

Consider the polynomial $f(T) = T^{28} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 28T^{27}$. If $p \nmid 28$, all roots of f are simple; since

$$f(T) = \prod_{d|28} \Phi_d(T),$$

all roots of Φ_{28} are simple when $p \nmid 28$. By (B14.3) $p \equiv 1 \pmod{28}$, hence $p \nmid 28$.

Setting $t \equiv X Y^{-1} \pmod{p}$ and using (B14.5), we obtain

$$\Phi_{28}(t) \equiv 0 \pmod{p}, \quad \Phi'_{28}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{28}(X/Y)) = 1. \quad (\text{B14.6})$$

Since $p \nmid Y$ and $p \nmid (X^2 + Y^2)$, from (B14.2) and (B14.6) it follows that

$$v_p(X^{14} + Y^{14}) = 1. \quad (\text{B14.7})$$

B14.5. Contradiction with the right-hand side

From (B14.0) we have

$$v_p(X^{14} + Y^{14}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B14.7) yields

$$v_p(X^{14} + Y^{14}) = 1.$$

Contradiction.

Hence, (B14.0) has no solutions.

B14.6. Conclusion for the case $g = 14$

- 2-adics (mod 8) exclude the “both odd” branch; the remaining case is mixed parity and C is odd (B14.1).
- The cyclotomic factorization (B14.2) separates the factors $X^2 + Y^2$ and $\Phi_{28}(X/Y)$ (independent with respect to odd primes).
- By Zsigmondy there exists a primitive prime $p \equiv 1 \pmod{28}$ with $\text{ord}_p(X/Y) = 28$, such that p divides $\Phi_{28}(X/Y)$ but does not divide $X^2 + Y^2$ (B14.3)–(B14.5).
- The simplicity of a root of Φ_{28} when $p \nmid 28$ gives $v_p = 1$ (B14.6), contradicting the right-hand side C^z for $z \geq 3$.

Therefore, for $g = 14$ the Beal equation has no solutions.

Appendix B_15. Complete analysis of the case $g = 15$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 15$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 15u, \quad y = 15v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ the equation becomes

$$X^{15} + Y^{15} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B15.0})$$

The goal is to derive a contradiction from (B15.0).

B15.1. Cyclotomic factorization and gcd control

We have the identity

$$X^{15} + Y^{15} = (X + Y) Q_{15}(X, Y),$$

where

$$Q_{15}(X, Y) = X^{14} - X^{13}Y + X^{12}Y^2 - \dots - XY^{13} + Y^{14} = Y^{14} \Phi_{30}\left(\frac{X}{Y}\right), \quad (\text{B15.1})$$

and $\Phi_{30}(T) = \frac{T^{30}-1}{(T^{15}-1)(T^{10}-1)(T^6-1)\dots}$ is the cyclotomic polynomial of order 30.

Classical fact:

$$\gcd(X + Y, Q_{15}(X, Y)) \mid 15. \quad (\text{B15.2})$$

In particular, any odd prime $p \neq 3, 5$ dividing $X^{15} + Y^{15}$ divides exactly one of the factors in (B15.1); its p -adic valuation on the left-hand side equals the valuation on the corresponding factor.

B15.2. Primitive divisor for the sum of fifteenth powers

Reminder: a prime p is called a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n) \quad \text{and} \quad p \nmid (X^m \pm Y^m) \quad \forall 1 \leq m < n$$

(the sign is fixed). For odd $n \geq 3$ and $\gcd(X, Y) = 1$, by the Bang–Zsigmondy theorem (strengthened to BHV) a primitive divisor of $X^n + Y^n$ exists with the sole exception $(n, X, Y) = (3, 2, 1)$ (and its symmetric case). For $n = 15$ there are no exceptions.

Lemma B15.1 (Zsigmondy for $n = 15$):

There exists an odd prime p such that

$$p \mid (X^{15} + Y^{15}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B15.3})$$

and moreover,

$$\text{ord}_p\left(\frac{X}{Y}\right) = 30, \quad p \equiv 1 \pmod{30}, \quad p \neq 3, 5. \quad (\text{B15.4})$$

Comment. A primitive divisor of a sum of odd degree divides precisely the “cyclotomic” factor $\Phi_{30}(X/Y)$; here $2n = 30$.

From (B15.2) and $p \nmid (X + Y)$ it follows that such a p divides exactly the cyclotomic factor:

$$p \mid Q_{15}(X, Y) = Y^{14} \Phi_{30}\left(\frac{X}{Y}\right), \quad \text{and since } p \nmid Y, \text{ we have } p \mid \Phi_{30}\left(\frac{X}{Y}\right). \quad (\text{B15.5})$$

B15.3. Uniqueness of the p -adic valuation (simple root of Φ_{30})

Consider the polynomial $f(T) = T^{30} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 30T^{29}$. If $p \nmid 30$ and $t \in \mathbb{F}_p^\times$ is a root of f , then $f'(t) \neq 0$; hence all roots of f are simple. Since

$$f = \prod_{d \mid 30} \Phi_d = \Phi_1 \Phi_2 \Phi_3 \Phi_5 \Phi_6 \Phi_{10} \Phi_{15} \Phi_{30},$$

every root of Φ_{30} is simple when $p \nmid 30$.

In our case (B15.4) guarantees $p \equiv 1 \pmod{30} \Rightarrow p \nmid 30$. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B15.5), we obtain

$$\Phi_{30}(t) \equiv 0 \pmod{p}, \quad \Phi'_{30}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{30}(X/Y)) = 1. \quad (\text{B15.6})$$

Since $p \nmid (X + Y)$ and $p \nmid Y$, from (B15.1) and (B15.6) it follows that

$$v_p(X^{15} + Y^{15}) = 1. \quad (\text{B15.7})$$

B15.4. Contradiction with the right-hand side

From (B15.0) we have

$$v_p(X^{15} + Y^{15}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B15.7) gives

$$v_p(X^{15} + Y^{15}) = 1.$$

Contradiction.

Hence, (B15.0) has no solutions.

B15.5. Conclusion for the case $g = 15$

- The cyclotomic factorization (B15.1) separates the linear factor $X + Y$ and the “cyclotomic” factor $Y^{14}\Phi_{30}(X/Y)$; by (B15.2) odd primes divide exactly one of them.
- By Zsigmondy there exists a primitive prime p with $p \nmid (X + Y)$, $\text{ord}_p(X/Y) = 30$, $p \equiv 1 \pmod{30}$ (Lemma B15.1); hence $p \mid \Phi_{30}(X/Y)$.
- The simplicity of a root of Φ_{30} when $p \nmid 30$ gives $v_p = 1$ (B15.6), which contradicts the right-hand side C^z for $z \geq 3$.

Therefore, for $g = 15$ the Beal equation has no solutions.

Appendix B_16. Complete analysis of the case $g = 16$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 16$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 16u, \quad y = 16v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ we have

$$X^{16} + Y^{16} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B16.0})$$

The goal is to derive a contradiction from (B16.0).

B16.1. 2-adic analysis: excluding the “both odd” branch and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence $X^4 \equiv Y^4 \equiv 1 \pmod{16}$, and therefore $X^{16} \equiv Y^{16} \equiv 1 \pmod{16}$, so

$$X^{16} + Y^{16} \equiv 2 \pmod{16} \Rightarrow v_2(X^{16} + Y^{16}) = 1.$$

For $z \geq 3$ this is impossible: if C is even, then $v_2(C^z) \geq 3$; if C is odd, then $C^z \equiv 1 \pmod{8}$. Hence the “both odd” branch is excluded.

Since $\gcd(X, Y) = 1$, the “both even” branch is impossible. Thus only the mixed parity remains: exactly one of X, Y is even and the other odd. Then the sum is odd; in particular

$$\boxed{C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3.} \quad (\text{B16.1})$$

B16.2. Cyclotomic factorization of $X^{16} + Y^{16}$

The general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right)$$

gives for $n = 16$ the single factor $d = 32$ (since $32 \mid 2n$ and $32 \nmid n$), i.e.

$$X^{16} + Y^{16} = Y^{16} \Phi_{32}\left(\frac{X}{Y}\right). \quad (\text{B16.2})$$

Hence any odd prime $p \nmid Y$ dividing $X^{16} + Y^{16}$ must divide $\Phi_{32}(X/Y)$.

B16.3. Primitive divisor for the sum of sixteenth powers

The Bang–Zsigmondy theorem (in the BHV strengthening) applies to $X^{16} + Y^{16}$ with $\gcd(X, Y) = 1$: for $n = 16$ there are no exceptions, hence a primitive prime divisor exists.

Lemma B16.1 (Zsigmondy for $n = 16$):

There exists an odd prime p such that

$$p \mid (X^{16} + Y^{16}), \quad p \nmid XY, \quad p \nmid (X + Y), \quad (\text{B16.3})$$

and

$$\text{ord}_p\left(\frac{X}{Y}\right) = 32, \quad p \equiv 1 \pmod{32}. \quad (\text{B16.4})$$

Comments. (i) Primitivity means that p divides none of the $X^m + Y^m$ for $m < 16$; this is equivalent to $p \mid \Phi_{32}(X/Y)$. (ii) If $p \mid X + Y$, then $\frac{X}{Y} \equiv -1 \pmod{p}$ would have order 2, contradicting (B16.4).

From (B16.2)–(B16.4) and $\gcd(X, Y) = 1$ we conclude:

$$p \mid \Phi_{32}\left(\frac{X}{Y}\right), \quad p \nmid Y. \quad (\text{B16.5})$$

B16.4. Uniqueness of the p -adic valuation (simple root of Φ_{32})

Consider the polynomial $f(T) = T^{32} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 32T^{31}$. If $p \nmid 32$ and $t \in \mathbb{F}_p^\times$ is a root of f , then $f'(t) \neq 0$; hence all roots of f are simple. Since

$$f = \prod_{d|32} \Phi_d, \quad \text{in particular, the roots of } \Phi_{32} \text{ are simple when } p \nmid 32.$$

From (B16.4) we have $p \equiv 1 \pmod{32} \Rightarrow p \nmid 32$. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B16.5), we obtain

$$\Phi_{32}(t) \equiv 0 \pmod{p}, \quad \Phi'_{32}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{32}(X/Y)) = 1. \quad (\text{B16.6})$$

Since $p \nmid Y$, it follows from (B16.2) and (B16.6) that

$$v_p(X^{16} + Y^{16}) = 1. \quad (\text{B16.7})$$

B16.5. Contradiction with the right-hand side

From (B16.0) we have

$$v_p(X^{16} + Y^{16}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B16.7) gives

$$v_p(X^{16} + Y^{16}) = 1.$$

Contradiction.

Hence, (B16.0) has no solutions.

B16.6. Conclusion for the case $g = 16$

- The 2-adic analysis excludes the “both odd” branch; the mixed parity remains and C is odd (B16.1).
- The cyclotomic factorization yields the precise form $X^{16} + Y^{16} = Y^{16}\Phi_{32}(X/Y)$ (B16.2).
- By Zsigmondy there exists a primitive $p \equiv 1 \pmod{32}$ with $\text{ord}_p(X/Y) = 32$, and $p \nmid (X + Y)$, $p \nmid XY$ (B16.3)–(B16.5).
- The simplicity of a root of Φ_{32} when $p \nmid 32$ gives $v_p = 1$ (B16.6), contradicting the right-hand side C^z for $z \geq 3$.

Thus, for $g = 16$ the Beal equation has no solutions.

Appendix B_17. Complete analysis of the case $g = 17$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 17$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 17u, \quad y = 17v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ the equation rewrites as

$$X^{17} + Y^{17} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B17.0})$$

The goal is to derive a contradiction from (B17.0).

B17.1. Cyclotomic factorization and gcd control

We have the identity

$$X^{17} + Y^{17} = (X + Y)Q_{17}(X, Y),$$

where

$$Q_{17}(X, Y) = X^{16} - X^{15}Y + X^{14}Y^2 - \dots - XY^{15} + Y^{16} = Y^{16}\Phi_{34}\left(\frac{X}{Y}\right), \quad (\text{B17.1})$$

and $\Phi_{34}(T) = T^{16} - T^{15} + T^{14} - \dots - T + 1$ is the cyclotomic polynomial of order 34.

Classical gcd fact:

$$\gcd(X + Y, Q_{17}(X, Y)) \mid 17. \quad (\text{B17.2})$$

Hence every odd prime $p \neq 17$ dividing $X^{17} + Y^{17}$ divides exactly one of the factors in (B17.1), and its p -adic valuation on the left-hand side equals the valuation on the corresponding factor.

B17.2. Primitive divisor for the sum of seventeenth powers

Reminder: a prime p is a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n) \quad \text{and} \quad p \nmid (X^m \pm Y^m) \quad \forall 1 \leq m < n$$

(with the sign fixed). For odd $n \geq 3$ and $\gcd(X, Y) = 1$, by the Bang–Zsigmondy theorem (strengthened to BHV) a primitive divisor of $X^n + Y^n$ exists with the sole exception $(n, X, Y) = (3, 2, 1)$ (and its symmetric case). For $n = 17$ there are no exceptions.

Lemma B17.1 (Zsigmondy for $n = 17$):

Let $\gcd(X, Y) = 1$. Then there exists a prime p such that

$$p \mid (X^{17} + Y^{17}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B17.3})$$

and moreover,

$$\text{ord}_p\left(\frac{X}{Y}\right) = 34, \quad p \equiv 1 \pmod{34}, \quad p \neq 17. \quad (\text{B17.4})$$

Comments. (i) Primitivity means that p divides none of the $X^m + Y^m$ with $m < 17$; this is equivalent to $p \mid \Phi_{34}(X/Y)$. (ii) If $p \mid X + Y$, then $\frac{X}{Y} \equiv -1 \pmod{p}$ would have order 2, contradicting (B17.4).

From (B17.2) and $p \nmid (X + Y)$ it follows immediately that our p divides exactly the cyclotomic factor,

$$p \mid Q_{17}(X, Y) = Y^{16}\Phi_{34}\left(\frac{X}{Y}\right), \quad \text{and since } p \nmid Y, \quad \text{we have } p \mid \Phi_{34}\left(\frac{X}{Y}\right). \quad (\text{B17.5})$$

B17.3. Uniqueness of the p -adic valuation (simplicity of a root of Φ_{34})

Consider the polynomial $f(T) = T^{34} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 34T^{33}$. If $p \nmid 34$ and $t \in \mathbb{F}_p^\times$ is a root of f , then $f'(t) \neq 0$; i.e. all roots of f are simple. Since

$$f = \prod_{d \mid 34} \Phi_d, \quad \text{in particular the roots of } \Phi_{34} \text{ are simple when } p \nmid 34.$$

From (B17.4) we have $p \equiv 1 \pmod{34} \Rightarrow p \nmid 34$. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B17.5), we obtain

$$\Phi_{34}(t) \equiv 0 \pmod{p}, \quad \Phi'_{34}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{34}(X/Y)) = 1. \quad (\text{B17.6})$$

Since $p \nmid (X + Y)$ and $p \nmid Y$, it follows from (B17.1) and (B17.6) that

$$v_p(X^{17} + Y^{17}) = 1. \quad (\text{B17.7})$$

B17.4. Contradiction with the right-hand side

From (B17.0) we have

$$v_p(X^{17} + Y^{17}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B17.7) gives

$$v_p(X^{17} + Y^{17}) = 1.$$

Contradiction.

Therefore, (B17.0) has no solutions.

B17.5. Conclusion for the case $g = 17$

- The cyclotomic factorization (B17.1) separates the linear factor $X + Y$ and the “cyclotomic” one $Y^{16}\Phi_{34}(X/Y)$; by (B17.2) odd primes divide exactly one of them.
- By Zsigmondy there exists a primitive p with $p \nmid (X + Y)$, $\text{ord}_p(X/Y) = 34$, $p \equiv 1 \pmod{34}$ (Lemma B17.1); hence $p \mid \Phi_{34}(X/Y)$.
- The simplicity of a root of Φ_{34} when $p \nmid 34$ gives $v_p = 1$ (B17.6), which contradicts the right-hand side C^z for $z \geq 3$.

Thus, for $g = 17$ the Beal equation has no solutions.

Appendix B_18. Complete analysis of the case $g = 18$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 18$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 18u, \quad y = 18v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ the equation becomes

$$X^{18} + Y^{18} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B18.0})$$

The goal is to derive a contradiction from (B18.0).

B18.1. 2-adic analysis: exclusion of the “both odd” branch and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence $X^{18} \equiv Y^{18} \equiv 1 \pmod{8}$ and

$$X^{18} + Y^{18} \equiv 2 \pmod{8} \implies v_2(X^{18} + Y^{18}) = 1.$$

For $z \geq 3$ the right-hand side yields either $v_2(C^z) \geq 3$ (if C is even), or $C^z \equiv 1 \pmod{8}$ (if C is odd)—a contradiction. Therefore the “both odd” branch is impossible.

Since $\gcd(X, Y) = 1$, the “both even” branch is excluded. What remains is mixed parity: exactly one of X, Y is even and the other is odd; then the sum is odd, in particular

$$\boxed{C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3.} \quad (\text{B18.1})$$

B18.2. Cyclotomic factorization of $X^{18} + Y^{18}$

The general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right)$$

gives for $n = 18$ the set $d \in \{4, 12, 36\}$ (divisors of 36 not dividing 18). Hence

$$X^{18} + Y^{18} = Y^{18} \Phi_4\left(\frac{X}{Y}\right) \Phi_{12}\left(\frac{X}{Y}\right) \Phi_{36}\left(\frac{X}{Y}\right), \quad (\text{B18.2})$$

where

$$\Phi_4(T) = T^2 + 1, \quad \Phi_{12}(T) = T^4 - T^2 + 1, \quad \Phi_{36}(T) = T^{12} - T^6 + 1. \quad (\text{B18.3})$$

B18.3. Primitive divisor for the sum of eighteenth powers

The Bang–Zsigmondy theorem (in the BHV strengthening) applies to $X^{18} + Y^{18}$ under $\gcd(X, Y) = 1$: for $n = 18$ there are no exceptions, hence a primitive prime divisor exists.

Lemma B18.1 (Zsigmondy for $n = 18$):

There exists an odd prime p such that

$$p \mid (X^{18} + Y^{18}), \quad p \nmid XY,$$

and moreover

$$\text{ord}_p\left(\frac{X}{Y}\right) = 36, \quad p \equiv 1 \pmod{36}.$$

Comments:

- (i) Primitivity with respect to $n = 18$ means that p divides $\Phi_{36}(X/Y)$ and does not divide the “earlier” factors Φ_4, Φ_{12} (their orders 4 and 12 would give $\text{ord}_p(X/Y) \in \{4, 12\}$, contradicting (B18.5)).
- (ii) From (B18.5) it follows automatically that $p \nmid (X + Y)$, otherwise the order would be 2.

Thus, from (B18.2)–(B18.5) and $\gcd(X, Y) = 1$ we obtain:

$$p \mid \Phi_{36}\left(\frac{X}{Y}\right), \quad p \nmid Y, \quad p \nmid \Phi_4\left(\frac{X}{Y}\right), \quad p \nmid \Phi_{12}\left(\frac{X}{Y}\right). \quad (\text{B18.6})$$

B18.4. Uniqueness of the p -adic valuation (simplicity of a root of Φ_{36})

Consider $f(T) = T^{36} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 36 T^{35}$. When $p \nmid 36$, all roots of f are simple; since

$$f = \prod_{d|36} \Phi_d(T),$$

in particular the roots of Φ_{36} are simple when $p \nmid 36$. By (B18.5) $p \equiv 1 \pmod{36} \Rightarrow p \nmid 36$.

Setting $t \equiv XY^{-1} \pmod{p}$ and using (B18.6), we obtain

$$\Phi_{36}(t) \equiv 0 \pmod{p}, \quad \Phi'_{36}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{36}(X/Y)) = 1. \quad (\text{B18.7})$$

Since $p \nmid Y$, it follows from (B18.2) and (B18.7) that

$$v_p(X^{18} + Y^{18}) = 1. \quad (\text{B18.8})$$

B18.5. Contradiction with the right-hand side

From (B18.0) we have

$$v_p(X^{18} + Y^{18}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B18.8) gives

$$v_p(X^{18} + Y^{18}) = 1.$$

Contradiction.

Therefore, (B18.0) has no solutions.

B18.6. Conclusion for the case $g = 18$

- The 2-adic analysis forbids the “both odd” branch; what remains is mixed parity and C is odd (B18.1).
- The cyclotomic factorization (B18.2) isolates the order-distinct factors $\Phi_4, \Phi_{12}, \Phi_{36}$.
- By Zsigmondy there exists a primitive prime $p \equiv 1 \pmod{36}$ with $\text{ord}_p(X/Y) = 36$ that divides $\Phi_{36}(X/Y)$ and divides neither $\Phi_4, \Phi_{12}, XY, X + Y$ (B18.4–B18.6).
- The simplicity of a root of Φ_{36} when $p \nmid 36$ yields $v_p = 1$ (B18.7), which contradicts the right-hand side C^z for $z \geq 3$.

Thus, for $g = 18$ the Beal equation has no solutions.

Appendix B19. Complete analysis of the case $g = 19$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 19$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 19u, \quad y = 19v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u, Y := B^v$ the equation becomes

$$X^{19} + Y^{19} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B19.0})$$

The goal is to derive a contradiction from (B19.0).

B19.1. Cyclotomic factorization and gcd control

We have

$$X^{19} + Y^{19} = (X + Y)Q_{19}(X, Y),$$

where

$$Q_{19}(X, Y) = X^{18} - X^{17}Y + X^{16}Y^2 - \dots - XY^{17} + Y^{18} = Y^{18}\Phi_{38}\left(\frac{X}{Y}\right), \quad (\text{B19.1})$$

and $\Phi_{38}(T) = T^{18} - T^{17} + T^{16} - \dots - T + 1$ is the cyclotomic polynomial of order 38.

A classical gcd fact:

$$\gcd(X + Y, Q_{19}(X, Y)) \mid 19. \quad (\text{B19.2})$$

Consequently, any odd prime $p \neq 19$ dividing $X^{19} + Y^{19}$ divides exactly one of the factors in (B19.1), and its p -adic valuation on the left-hand side equals the valuation on the corresponding factor.

B19.2. A primitive divisor for the sum of nineteenth powers

A prime p is called a primitive divisor of $X^n \pm Y^n$ if

$$p \mid (X^n \pm Y^n), \quad p \nmid (X^m \pm Y^m), \quad \forall 1 \leq m < n$$

(with the sign fixed). For odd $n \geq 3$ and $\gcd(X, Y) = 1$, by Bang–Zsigmondy (BHV strengthening) a primitive divisor of $X^n + Y^n$ exists except for the unique case $(n, X, Y) = (3, 2, 1)$. For $n = 19$ there are no exceptions.

Lemma B19.1 (Zsigmondy for $n = 19$)

There exists a prime p such that

$$p \mid (X^{19} + Y^{19}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B19.3})$$

and moreover

$$\text{ord}_p\left(\frac{X}{Y}\right) = 38, \quad p \equiv 1 \pmod{38}, \quad p \neq 19. \quad (\text{B19.4})$$

Comment: A primitive divisor of a sum of odd powers divides $\Phi_{2n}(X/Y)$; here $2n = 38$, which yields (B19.4).

From (B19.2) and $p \nmid (X + Y)$ it follows that such a p divides precisely the “cyclotomic” factor:

$$p \mid Q_{19}(X, Y) = Y^{18}\Phi_{38}\left(\frac{X}{Y}\right), \quad \text{and since } p \nmid Y, \text{ we have } p \mid \Phi_{38}\left(\frac{X}{Y}\right). \quad (\text{B19.5})$$

B19.3. Uniqueness of the p -adic valuation (simplicity of a root of Φ_{38})

Consider $f(T) = T^{38} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 38T^{37}$. When $p \nmid 38$, all roots of f are simple; since

$$f = \prod_{d \mid 38} \Phi_d = \Phi_1 \Phi_2 \Phi_{19} \Phi_{38},$$

the roots of Φ_{38} are simple when $p \nmid 38$.

In our case (B19.4) gives $p \equiv 1 \pmod{38} \Rightarrow p \nmid 38$. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B19.5), we obtain

$$\Phi_{38}(t) \equiv 0 \pmod{p}, \quad \Phi'_{38}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{38}(X/Y)) = 1. \quad (\text{B19.6})$$

Since $p \nmid (X + Y)$ and $p \nmid Y$, from (B19.1) and (B19.6) we conclude

$$v_p(X^{19} + Y^{19}) = 1. \quad (\text{B19.7})$$

B19.4. Contradiction with the right-hand side

From (B19.0) it follows that

$$v_p(X^{19} + Y^{19}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B19.7) gives

$$v_p(X^{19} + Y^{19}) = 1.$$

Contradiction.

Therefore, (B19.0) has no solutions.

B19.5. Conclusion for the case $g = 19$

- The cyclotomic factorization (B19.1) separates the linear factor $X + Y$ and the “cyclotomic” factor $Y^{18}\Phi_{38}(X/Y)$; by (B19.2) odd primes divide exactly one of them.
- By Zsigmondy there exists a primitive p with $p \nmid (X + Y)$, $\text{ord}_p(X/Y) = 38$, $p \equiv 1 \pmod{38}$ (Lemma B19.1); then $p \mid \Phi_{38}(X/Y)$.
- The simplicity of a root of Φ_{38} when $p \nmid 38$ yields $v_p = 1$ (B19.6), which contradicts the right-hand side C^z for $z \geq 3$.

Thus, for $g = 19$ the Beal equation has no solutions.

Appendix B₂₀. Complete analysis of the case $g = 20$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 20$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 20u, \quad y = 20v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ the equation becomes

$$X^{20} + Y^{20} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B20.0})$$

The goal is to derive a contradiction from (B20.0).

B20.1. 2-adics: exclusion of the “both odd” branch and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence $X^4 \equiv Y^4 \equiv 1 \pmod{16}$, and therefore $X^{20} \equiv Y^{20} \equiv 1 \pmod{16}$. Consequently

$$X^{20} + Y^{20} \equiv 2 \pmod{16} \Rightarrow v_2(X^{20} + Y^{20}) = 1,$$

which is incompatible with the right-hand side for $z \geq 3$: if C is even, then $v_2(C^z) \geq 3$; if C is odd, then $C^z \equiv 1 \pmod{8}$. Thus the “both odd” branch is excluded.

Since $\gcd(X, Y) = 1$, the “both even” branch is impossible. The remaining case is mixed parity: exactly one of X, Y is even, the other is odd. Then the sum is odd; in particular

$$\boxed{C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3.} \quad (\text{B20.1})$$

B20.2. Cyclotomic factorization of $X^{20} + Y^{20}$

The general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right)$$

gives for $n = 20$ the set $d \in \{8, 40\}$ (divisors of 40 not dividing 20). Hence

$$X^{20} + Y^{20} = Y^{20} \Phi_8\left(\frac{X}{Y}\right) \Phi_{40}\left(\frac{X}{Y}\right), \quad \Phi_8(T) = T^4 + 1, \quad \deg \Phi_{40} = 16. \quad (\text{B20.2})$$

As usual,

$$\gcd\left(\Phi_8\left(\frac{X}{Y}\right), \Phi_{40}\left(\frac{X}{Y}\right)\right) = 1$$

with respect to odd primes. In particular, any odd prime p divides exactly one of the factors in (B20.2).

B20.3. A primitive divisor for the sum of twentieth powers

Bang–Zsigmondy (in the BHV strengthening) applies to $X^{20} + Y^{20}$ under $\gcd(X, Y) = 1$: for $n = 20 > 2$ there are no exceptions, hence a primitive prime divisor exists.

Lemma B20.1 (Zsigmondy for $n = 20$):

There exists an odd prime p such that

$$p \mid (X^{20} + Y^{20}), \quad p \nmid XY, \quad (\text{B20.3})$$

and

$$\text{ord}_p\left(\frac{X}{Y}\right) = 40, \quad p \equiv 1 \pmod{40}. \quad (\text{B20.4})$$

Comments:

1. For a prime divisor of $X^n + Y^n$, the order $\text{ord}_p(X/Y)$ divides $2n$ and does not divide n (since $(X/Y)^n \equiv -1$), while primitiveness excludes divisors for smaller $m < n$; hence $\text{ord}_p = 2n = 40$.
2. If $p \mid \Phi_8(X/Y)$, then $\text{ord}_p(X/Y) = 8$, contradicting (B20.4). Therefore p divides precisely $\Phi_{40}(X/Y)$.

Thus, from (B20.2)–(B20.4) and $\gcd(X, Y) = 1$ we obtain

$$p \mid \Phi_{40}\left(\frac{X}{Y}\right), \quad p \nmid \Phi_8\left(\frac{X}{Y}\right), \quad p \nmid Y, \quad p \nmid (X + Y). \quad (\text{B20.5})$$

B20.4. Uniqueness of the p -adic valuation (simplicity of a root of Φ_{40})

Consider $f(T) = T^{40} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 40T^{39}$.

When $p \nmid 40$, all roots of f are simple; since

$$f = \prod_{d|40} \Phi_d(T),$$

the roots of Φ_{40} are in particular simple when $p \nmid 40$. By (B20.4) we have $p \equiv 1 \pmod{40} \Rightarrow p \nmid 40$.

Setting $t \equiv XY^{-1} \pmod{p}$ and using (B20.5), we get

$$\Phi_{40}(t) \equiv 0 \pmod{p}, \quad \Phi'_{40}(t) \not\equiv 0 \pmod{p} \implies v_p\left(\Phi_{40}\left(\frac{X}{Y}\right)\right) = 1. \quad (\text{B20.6})$$

Since $p \nmid Y$, from (B20.2) and (B20.6) we conclude

$$v_p(X^{20} + Y^{20}) = 1. \quad (\text{B20.7})$$

B20.5. Contradiction with the right-hand side

From (B20.0) it follows that

$$v_p(X^{20} + Y^{20}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B20.7) gives

$$v_p(X^{20} + Y^{20}) = 1.$$

Contradiction.

Therefore, (B20.0) has no solutions.

B20.6. Conclusion for the case $g = 20$

- The 2-adic argument excludes the “both odd” branch; the mixed parity remains and C is odd (B20.1).
- The cyclotomic factorization (B20.2) yields independent factors $\Phi_8(X/Y)$ and $\Phi_{40}(X/Y)$.
- By Zsigmondy there exists a primitive prime $p \equiv 1 \pmod{40}$ with $\text{ord}_p(X/Y) = 40$; hence p divides $\Phi_{40}(X/Y)$ but not $\Phi_8(X/Y)$, and also $p \nmid XY$ and $p \nmid (X + Y)$ (B20.3)–(B20.5).
- The simplicity of a root of Φ_{40} when $p \nmid 40$ gives $v_p = 1$ (B20.6), which contradicts the right-hand side C^z for $z \geq 3$.

Thus, for $g = 20$ the Beal equation has no solutions.

Appendix B_{21} . Complete analysis of the case $g = 21$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 21$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 21u, \quad y = 21v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u, Y := B^v$ the equation becomes

$$X^{21} + Y^{21} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B21.0})$$

The goal is to derive a contradiction from (B21.0).

B21.1. Cyclotomic factorization and gcd control

We have

$$X^{21} + Y^{21} = (X + Y)Q_{21}(X, Y),$$

where

$$Q_{21}(X, Y) = X^{20} - X^{19}Y + \cdots - XY^{19} + Y^{20}. \quad (\text{B21.1})$$

A standard fact (from properties of homogenizations of cyclotomic polynomials) is that

$$\gcd(X + Y, Q_{21}(X, Y)) \mid 21. \quad (\text{B21.2})$$

In particular, any odd prime $p \neq 3, 7$ dividing $X^{21} + Y^{21}$ divides exactly one of the factors in (B21.1); its p -adic valuation on the left-hand side equals the valuation on the corresponding factor.

An equivalent “one-variable” form: when $Y \not\equiv 0 \pmod{p}$ and $t \equiv XY^{-1} \pmod{p}$,

$$X^{21} + Y^{21} \equiv Y^{21} (t^{21} + 1) = Y^{21} \cdot \Phi_2(t) \cdot \Phi_{42}(t) \cdot R(t),$$

where R is the product of Φ_d over the divisors of $2 \cdot 21$ with $d < 42$ and $d \nmid 21$, and $\Phi_2(t) = t + 1$. For our purposes it is crucial that if $\text{ord}_p(t) = 42$, then p divides only $\Phi_{42}(t)$, dividing neither $\Phi_2(t)$ nor any $\Phi_d(t)$ with $d < 42$.

B21.2. A primitive divisor for the sum of 21-st powers

By the Bang–Zsigmondy theorem (in the BHV strengthening): for odd $n \geq 3$ and $\gcd(X, Y) = 1$, the number $X^n + Y^n$ has a primitive prime divisor, with the unique exception $(n, X, Y) = (3, 2, 1)$. For $n = 21$ there are no exceptions.

Lemma B21.1 (Zsigmondy for $n = 21$):

There exists an odd prime p such that

$$p \mid (X^{21} + Y^{21}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B21.3})$$

and moreover

$$\text{ord}_p\left(\frac{X}{Y}\right) = 42, \quad p \equiv 1 \pmod{42}, \quad p \neq 3, 7. \quad (\text{B21.4})$$

Comment: Primitiveness for a sum of odd powers is equivalent to $\frac{X}{Y}$ having order $2n$ and annihilating Φ_{2n} , while not annihilating Φ_2 (which would correspond to $X + Y \equiv 0$) nor any Φ_d with $d < 2n$.

From (B21.2) and $p \nmid (X + Y)$ we get that such a p divides precisely the “cyclotomic” factor in (B21.1); equivalently,

$$p \mid \Phi_{42}\left(\frac{X}{Y}\right), \quad p \nmid Y. \quad (\text{B21.5})$$

B21.3. Uniqueness of the p -adic valuation (simplicity of a root of Φ_{42})

Consider $f(T) = T^{42} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 42T^{41}$.

When $p \nmid 42$, all roots of f are simple; since

$$f = \prod_{d \mid 42} \Phi_d(T),$$

the roots of Φ_{42} are simple when $p \nmid 42$.

By (B21.4) $p \equiv 1 \pmod{42} \Rightarrow p \nmid 42$. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B21.5), we obtain

$$\Phi_{42}(t) \equiv 0 \pmod{p}, \quad \Phi'_{42}(t) \not\equiv 0 \pmod{p} \implies v_p\left(\Phi_{42}\left(\frac{X}{Y}\right)\right) = 1. \quad (\text{B21.6})$$

Since $p \nmid Y$ and $p \nmid (X + Y)$, from the factorization (B21.1) and (B21.6) we conclude that

$$v_p(X^{21} + Y^{21}) = 1. \quad (\text{B21.7})$$

B21.4. Contradiction with the right-hand side

From (B21.0) it follows that

$$v_p(X^{21} + Y^{21}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B21.7) gives $v_p(X^{21} + Y^{21}) = 1$. Contradiction.

Therefore, (B21.0) has no solutions.

B21.5. Conclusion for the case $g = 21$

- The cyclotomic factorization (B21.1) separates $X+Y$ and the “high-frequency” factor; by (B21.2) odd primes $p \neq 3, 7$ divide exactly one of them.
- By Zsigmondy there exists a primitive $p \equiv 1 \pmod{42}$ with $\text{ord}_p(X/Y) = 42$ and $p \nmid (X + Y), XY$ (Lemma B21.1); hence $p \mid \Phi_{42}(X/Y)$.
- The simplicity of a root of Φ_{42} when $p \nmid 42$ yields $v_p = 1$ (B21.6), which contradicts the right-hand side C^z for $z \geq 3$.

Thus, for $g = 21$ the Beal equation has no solutions. ■

Appendix B₂₂. Complete analysis of the case $g = 22$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 22$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ with

$$x = 22u, \quad y = 22v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ the equation takes the form

$$X^{22} + Y^{22} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B22.0})$$

The goal is to derive a contradiction from (B22.0).

B22.1. 2-adics: forbidding the “both odd” branch and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence $X^{22} \equiv Y^{22} \equiv 1 \pmod{8}$ and

$$X^{22} + Y^{22} \equiv 2 \pmod{8} \Rightarrow v_2(X^{22} + Y^{22}) = 1.$$

For $z \geq 3$ the right-hand side gives either $v_2(C^z) \geq 3$ (if C is even) or $C^z \equiv 1 \pmod{8}$ (if C is odd); a contradiction. Since $\gcd(X, Y) = 1$, the branch “both even” is impossible. Thus we are left with mixed parity: exactly one of X, Y is even and the other is odd; then the sum is odd, and

$$\boxed{C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3.} \quad (\text{B22.1})$$

B22.2. Cyclotomic factorization of $X^{22} + Y^{22}$

The general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right)$$

gives for $n = 22$ the divisors $d \in \{4, 44\}$ (from $\{1, 2, 4, 11, 22, 44\}$ we exclude $1, 2, 11, 22$). Hence

$$X^{22} + Y^{22} = Y^{22} \Phi_4\left(\frac{X}{Y}\right) \Phi_{44}\left(\frac{X}{Y}\right),$$

where

$$\Phi_4(T) = T^2 + 1, \quad \deg \Phi_{44} = 20. \quad (\text{B22.2})$$

For odd prime divisors we have $\gcd(\Phi_4(X/Y), \Phi_{44}(X/Y)) = 1$; consequently, any odd prime p divides exactly one of the factors in (B22.2).

B22.3. A primitive divisor for the sum of twenty-second powers

By the Bang–Zsigmondy theorem (BHV version): for $\gcd(X, Y) = 1$ and $n = 22 > 2$ there exists a primitive prime divisor of $X^{22} + Y^{22}$.

Lemma B22.1 (Zsigmondy for $n = 22$):

There exists an odd prime p such that

$$p \mid (X^{22} + Y^{22}), \quad p \nmid XY, \quad (\text{B22.3})$$

and

$$\text{ord}_p\left(\frac{X}{Y}\right) = 44, \quad p \equiv 1 \pmod{44}. \quad (\text{B22.4})$$

Comment: (i) For a divisor of $X^n + Y^n$ the order $\text{ord}_p(X/Y)$ divides $2n$ and does not divide n (since $(X/Y)^n \equiv -1$). Primitiveness rules out smaller exponents, hence $\text{ord}_p = 2n = 44$. (ii) If $p \mid \Phi_4(X/Y)$, then $\text{ord}_p(X/Y) = 4$, contradicting (B22.4). Also $p \nmid (X+Y)$ (otherwise the order would be 2).

From (B22.2)–(B22.4) and $\gcd(X, Y) = 1$ we obtain

$$p \mid \Phi_{44}\left(\frac{X}{Y}\right), \quad p \nmid \Phi_4\left(\frac{X}{Y}\right), \quad p \nmid Y, \quad p \nmid (X+Y). \quad (\text{B22.5})$$

B22.4. Uniqueness of the p -adic valuation (simplicity of a root of Φ_{44})

Consider $f(T) = T^{44} - 1$ over $\mathbb{F}_p$; $f'(T) = 44T^{43}$. If $p \nmid 44$ and $t \in \mathbb{F}_p^\times$ is a root of f , then $f'(t) \neq 0$; hence all roots of f are simple and, in particular, the roots of Φ_{44} are simple. From (B22.4) $p \equiv 1 \pmod{44} \Rightarrow p \nmid 44$.

Setting $t \equiv XY^{-1} \pmod{p}$ and using (B22.5), we have

$$\Phi_{44}(t) \equiv 0 \pmod{p}, \quad \Phi'_{44}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{44}(X/Y)) = 1. \quad (\text{B22.6})$$

Since $p \nmid Y$, from (B22.2) and (B22.6) we conclude that

$$v_p(X^{22} + Y^{22}) = 1. \quad (\text{B22.7})$$

B22.5. Contradiction with the right-hand side

From (B22.0) it follows that

$$v_p(X^{22} + Y^{22}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B22.7) gives

$$v_p(X^{22} + Y^{22}) = 1.$$

Contradiction.

Therefore, (B22.0) has no solutions.

B22.6. Conclusion for the case $g = 22$

- The 2-adic analysis rules out the “both odd” branch; we are left with mixed parity and C is odd (B22.1).
- The cyclotomic factorization (B22.2) yields the independent factors $\Phi_4(X/Y)$ and $\Phi_{44}(X/Y)$.
- By Zsigmondy there exists a primitive prime $p \equiv 1 \pmod{44}$ with $\text{ord}_p(X/Y) = 44$; hence p divides $\Phi_{44}(X/Y)$ but not $\Phi_4(X/Y)$, and $p \nmid XY$, $X + Y$ (B22.3)–(B22.5).
- The simplicity of a root of Φ_{44} when $p \nmid 44$ gives $v_p = 1$ (B22.6), contradicting the right-hand side C^z for $z \geq 3$.

Hence, for $g = 22$ the Beal equation has no solutions. ■

B23. Complete analysis of the case $g = 23$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and let $g = \gcd(x, y) = 23$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 23u, \quad y = 23v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ we obtain

$$X^{23} + Y^{23} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B23.0})$$

The goal is to derive a contradiction from (B23.0).

B23.1. Cyclotomic factorization and gcd control

We have

$$X^{23} + Y^{23} = (X + Y)Q_{23}(X, Y),$$

where

$$Q_{23}(X, Y) = X^{22} - X^{21}Y + X^{20}Y^2 - \dots - XY^{21} + Y^{22} = Y^{22}\Phi_{46}\left(\frac{X}{Y}\right). \quad (\text{B23.1})$$

Standard fact:

$$\gcd(X + Y, Q_{23}(X, Y)) \mid 23. \quad (\text{B23.2})$$

Consequently, any odd prime $p \neq 23$ dividing $X^{23} + Y^{23}$ divides exactly one of the factors $X + Y$ or $Q_{23}(X, Y)$.

B23.2. A primitive divisor of the sum of twenty-third powers

By the Bang–Zsigmondy theorem (BHV refinement): for odd $n \geq 3$ and $\gcd(X, Y) = 1$, the number $X^n + Y^n$ has a primitive prime divisor, with the sole exception $(n, X, Y) = (3, 2, 1)$. For $n = 23$ there are no exceptions.

Lemma (Zsigmondy for $n = 23$). There exists a prime p such that

$$p \mid (X^{23} + Y^{23}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B23.3})$$

and

$$\text{ord}_p\left(\frac{X}{Y}\right) = 46, \quad p \equiv 1 \pmod{46}, \quad p \neq 23. \quad (\text{B23.4})$$

Comment. Primitiveness for a sum of odd powers means that $\left(\frac{X}{Y}\right)^{23} \equiv -1 \pmod{p}$ and the order is $\text{ord}_p(X/Y) = 2 \cdot 23 = 46$; then p divides $\Phi_{46}(X/Y)$, but not $X + Y$ (otherwise the order would be 2).

From (B23.2) and $p \nmid (X + Y)$ it follows that such a p divides precisely the “cyclotomic” factor:

$$p \mid Q_{23}(X, Y) = Y^{22}\Phi_{46}\left(\frac{X}{Y}\right), \quad \text{and since } p \nmid Y, \quad \text{we have } p \mid \Phi_{46}\left(\frac{X}{Y}\right). \quad (\text{B23.5})$$

B23.3. Uniqueness of the p -adic valuation (simplicity of a root of Φ_{46})

Consider $f(T) = T^{46} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 46T^{45}$. If $p \nmid 46$ and $t \in \mathbb{F}_p^\times$ is a root of f , then $f'(t) \neq 0$; hence all roots of f are simple. Since

$$f = \prod_{d \mid 46} \Phi_d = \Phi_1 \Phi_2 \Phi_{23} \Phi_{46},$$

the roots of Φ_{46} are simple when $p \nmid 46$.

From (B23.4) we have $p \equiv 1 \pmod{46} \Rightarrow p \nmid 46$. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B23.5), we obtain

$$\Phi_{46}(t) \equiv 0 \pmod{p}, \quad \Phi'_{46}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{46}(X/Y)) = 1. \quad (\text{B23.6})$$

Since $p \nmid Y$ and $p \nmid (X + Y)$, from (B23.1) and (B23.6) we conclude that

$$v_p(X^{23} + Y^{23}) = 1. \quad (\text{B23.7})$$

B23.4. Contradiction with the right-hand side

From (B23.0):

$$v_p(X^{23} + Y^{23}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B23.7) yields

$$v_p(X^{23} + Y^{23}) = 1.$$

Contradiction.

Therefore, (B23.0) has no solutions.

B23.5. Conclusion for the case $g = 23$

- The cyclotomic factorization (B23.1) separates $X + Y$ and $Y^{22}\Phi_{46}(X/Y)$; by (B23.2) odd primes $p \neq 23$ divide exactly one of them.
- By Zsigmondy there exists a primitive $p \equiv 1 \pmod{46}$ with $\text{ord}_p(X/Y) = 46$ and $p \nmid (X + Y), XY$ (Lemma B23.1); hence $p \mid \Phi_{46}(X/Y)$.
- The simplicity of a root of Φ_{46} when $p \nmid 46$ gives $v_p = 1$ (B23.6), which contradicts the right-hand side C^z for $z \geq 3$.

Hence, for $g = 23$ the Beal equation has no solutions.

B24. Complete analysis of the case $g = 24$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 24$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 24u, \quad y = 24v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u, Y := B^v$ we obtain

$$X^{24} + Y^{24} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B24.0})$$

The goal is to derive a contradiction from (B24.0).

B24.1. 2-adic analysis: ruling out “both odd” and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence $X^4 \equiv Y^4 \equiv 1 \pmod{16}$. Since $24 \equiv 0 \pmod{4}$,

$$X^{24} \equiv Y^{24} \equiv 1 \pmod{16} \Rightarrow X^{24} + Y^{24} \equiv 2 \pmod{16} \Rightarrow v_2(X^{24} + Y^{24}) = 1.$$

For $z \geq 3$ the right-hand side gives either $v_2(C^z) \geq 3$ (if C is even) or $v_2(C^z) = 0$ (if C is odd) — a contradiction. Since $\gcd(X, Y) = 1$, the case “both even” is impossible. Therefore, exactly one of X, Y is even and the other odd, and the sum is odd:

$$C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3. \quad (\text{B24.1})$$

B24.2. Cyclotomic factorization of $X^{24} + Y^{24}$

By the general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right),$$

for $n = 24$ we get $d \in \{16, 48\}$. Hence

$$X^{24} + Y^{24} = Y^{24} \Phi_{16}\left(\frac{X}{Y}\right) \Phi_{48}\left(\frac{X}{Y}\right), \quad \Phi_{16}(T) = T^8 + 1, \quad \deg \Phi_{48} = 16. \quad (\text{B24.2})$$

If an odd $p \nmid 48$, then the roots of Φ_{16} and Φ_{48} in $\mathbb{F}_p$ are disjoint; consequently, such a p divides exactly one factor in (B24.2). In §B24.3 we choose $p \equiv 1 \pmod{48}$, and thus $p \nmid 48$.

B24.3. A prime from Φ_{48} via BHV for $X^{48} - Y^{48}$

Theorem (Bilu–Hanrot–Voutier). If $\gcd(X, Y) = 1$ and $n \geq 31$, then $X^n - Y^n$ has a primitive prime divisor. Applying this for $n = 48$, we obtain an odd prime p such that

$$p \mid (X^{48} - Y^{48}), \quad p \nmid (X^m - Y^m) \quad \forall m < 48, \quad p \nmid XY.$$

Therefore,

$$\text{ord}_p\left(\frac{X}{Y}\right) = 48, \quad p \equiv 1 \pmod{48}, \quad p \nmid 48. \quad (\text{B24.3})$$

Then $\left(\frac{X}{Y}\right)^{24} \equiv -1 \pmod{p}$, i.e.

$$X^{24} + Y^{24} \equiv 0 \pmod{p}.$$

More precisely,

$$p \mid \Phi_{48}\left(\frac{X}{Y}\right), \quad p \nmid \Phi_{16}\left(\frac{X}{Y}\right), \quad p \nmid (X + Y), \quad p \nmid Y. \quad (\text{B24.4})$$

B24.4. Uniqueness of the p -adic valuation (simple zero of Φ_{48})

Consider $f(T) = T^{48} - 1$ over $\mathbb{F}_p$. The derivative is $f'(T) = 48T^{47}$. From $p \nmid 48$ (see (B24.3)) it follows that $T^{48} - 1$ has no multiple roots; in particular, the roots of Φ_{48} are simple. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B24.4),

$$\Phi_{48}(t) \equiv 0, \quad \Phi'_{48}(t) \not\equiv 0 \pmod{p}.$$

Since Φ_{48} is monic and $p \nmid Y$, we get

$$v_p(\Phi_{48}(X/Y)) = 1, \tag{B24.5}$$

and, substituting into (B24.2),

$$v_p(X^{24} + Y^{24}) = 1. \tag{B24.6}$$

B24.5. Contradiction with the right-hand side

From (B24.0): $X^{24} + Y^{24} = C^z$. Since $p \mid X^{24} + Y^{24}$, we have $p \mid C$. Hence $v_p(C^z) \geq z \geq 3$, whereas (B24.6) yields $v_p(X^{24} + Y^{24}) = 1$. Contradiction.

Therefore, (B24.0) has no solutions.

B24.6. Conclusion for the case $g = 24$

- The 2-adic analysis rules out the branch “both odd”; the remaining case is mixed parity and C is odd (B24.1).
- Cyclotomic factorization (B24.2): the factors $\Phi_{16}(X/Y)$ and $\Phi_{48}(X/Y)$; for our prime $p \equiv 1 \pmod{48}$ they are separated.
- By BHV for $X^{48} - Y^{48}$ there exists $p \equiv 1 \pmod{48}$ with $\text{ord}_p(X/Y) = 48$; such a p divides precisely $\Phi_{48}(X/Y)$ and $X^{24} + Y^{24}$, with $p \nmid XY, X + Y$.
- The simple zero of Φ_{48} (since $p \nmid 48$) gives $v_p = 1$, which contradicts the right-hand side C^z for $z \geq 3$.

Hence, for $g = 24$ the Beal equation has no solutions. ■

B25. Complete analysis of the case $g = 25$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 25$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 25u, \quad y = 25v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u, Y := B^v$ we obtain

$$X^{25} + Y^{25} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B25.0}$$

The goal is to derive a contradiction from (B25.0).

B25.1. Cyclotomic factorization and gcd control

For odd n :

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d \left(\frac{X}{Y} \right).$$

For $n = 25$ (divisors of 50: 1, 2, 5, 10, 25, 50) we get

$$X^{25} + Y^{25} = Y^{25} \Phi_2 \left(\frac{X}{Y} \right) \Phi_{10} \left(\frac{X}{Y} \right) \Phi_{50} \left(\frac{X}{Y} \right), \quad (\text{B25.1})$$

where $\Phi_2(T) = T + 1$, $\Phi_{10}(T) = T^4 - T^3 + T^2 - T + 1$, $\deg \Phi_{50} = 20$.

Equivalently,

$$X^{25} + Y^{25} = (X + Y) Q_{25}(X, Y), \quad Q_{25} = Y^{24} \Phi_{10} \left(\frac{X}{Y} \right) \Phi_{50} \left(\frac{X}{Y} \right), \quad (\text{B25.2})$$

and, standardly,

$$\gcd \left(X + Y, \frac{X^{25} + Y^{25}}{X + Y} \right) = \gcd(X + Y, 25) \mid 25. \quad (\text{B25.3})$$

Remarks:

1. If an odd $p \mid (X^{25} + Y^{25})$, then $p \nmid X$ and $p \nmid Y$ (otherwise $X^{25} + Y^{25} \equiv X^{25}$ or $\equiv Y^{25} \not\equiv 0 \pmod{p}$). Therefore it is legitimate to work with $t := XY^{-1} \in \mathbb{F}_p^\times$.
2. For $p \nmid 50$ the polynomials $\Phi_2, \Phi_{10}, \Phi_{50}$ have pairwise disjoint sets of roots in $\mathbb{F}_p$ (since $T^{50} - 1$ has no multiple roots), and therefore the corresponding values are pairwise coprime. The case $p \mid 50$ will not be needed.

B25.2. Primitive divisor of the sum of 25th powers

By Bang–Zsigmondy (in the strengthened form of Bilu–Hanrot–Voutier): for $\gcd(X, Y) = 1$ and $n > 2$, the number $X^n + Y^n$ has a primitive prime divisor (the only exception for the sum is $(n, X, Y) = (3, 2, 1)$). For $n = 25$ there are no exceptions. Hence there exists an odd prime p such that

$$p \mid (X^{25} + Y^{25}), \quad p \nmid XY, \quad (\text{B25.4})$$

and, since $\left(\frac{X}{Y}\right)^{25} \equiv -1 \pmod{p}$,

$$\text{ord}_p \left(\frac{X}{Y} \right) \mid 50, \quad \text{ord}_p \left(\frac{X}{Y} \right) \nmid 25.$$

The order options 2, 10 are excluded by primitiveness:

$$\text{ord} = 2 \Rightarrow p \mid X + Y \quad (\text{the linear factor}), \quad \text{ord} = 10 \Rightarrow p \mid X^5 + Y^5 \quad (5 < 25).$$

Orders 5, 25 are impossible since $\left(\frac{X}{Y}\right)^{25} \equiv -1$. Hence

$$\text{ord}_p \left(\frac{X}{Y} \right) = 50, \quad p \equiv 1 \pmod{50}. \quad (\text{B25.5})$$

Then from (B25.1), with $\gcd(X, Y) = 1$ and $p \nmid 50$, it follows that

$$p \mid \Phi_{50} \left(\frac{X}{Y} \right), \quad p \nmid \Phi_2 \left(\frac{X}{Y} \right), \quad p \nmid \Phi_{10} \left(\frac{X}{Y} \right), \quad p \nmid Y. \quad (\text{B25.6})$$

B25.3. Uniqueness of the p -adic valuation (simple zero of Φ_{50})

Consider $f(T) = T^{50} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 50T^{49}$. From $p \equiv 1 \pmod{50}$ we get $p \nmid 50$, hence $T^{50} - 1$ has no multiple roots over $\mathbb{F}_p$; in particular, all roots of Φ_{50} are simple. For $t \equiv XY^{-1} \pmod{p}$ and using (B25.6),

$$\Phi_{50}(t) \equiv 0, \quad \Phi'_{50}(t) \not\equiv 0 \pmod{p}.$$

Since Φ_{50} is monic and $p \nmid Y$, we obtain

$$v_p(\Phi_{50}(X/Y)) = 1, \tag{B25.7}$$

and, substituting into (B25.1), we have

$$v_p(X^{25} + Y^{25}) = 1. \tag{B25.8}$$

B25.4. Contradiction with the right-hand side

From (B25.0): $X^{25} + Y^{25} = C^z$. Since $p \mid X^{25} + Y^{25}$, we have $p \mid C$. Hence $v_p(C^z) \geq z \geq 3$, whereas (B25.8) gives $v_p(X^{25} + Y^{25}) = 1$. Contradiction.

Therefore, (B25.0) has no solutions.

B25.5. Conclusion for the case $g = 25$

- Cyclotomic factorization (B25.1): three factors $\Phi_2, \Phi_{10}, \Phi_{50}$; for $p \nmid 50$ their roots in $\mathbb{F}_p$ are pairwise disjoint.
- By Zsigmondy–BHV there exists a primitive $p \equiv 1 \pmod{50}$ with $\text{ord}_p(X/Y) = 50$; then p divides precisely $\Phi_{50}(X/Y)$, but not Φ_2, Φ_{10} , and $p \nmid XY$.
- The simple zero of Φ_{50} for $p \nmid 50$ yields $v_p = 1$, which contradicts the right-hand side C^z for $z \geq 3$.

Hence, for $g = 25$ the Beal equation has no solutions. ■

B26. Complete analysis of the case $g = 26$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 26$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 26u, \quad y = 26v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u, Y := B^v$ we obtain

$$X^{26} + Y^{26} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B26.0}$$

The goal is to derive a contradiction from (B26.0).

B26.1. 2-adics: exclusion of “both odd” and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence

$$X^{26} \equiv Y^{26} \equiv 1 \pmod{8} \Rightarrow X^{26} + Y^{26} \equiv 2 \pmod{8} \Rightarrow v_2(X^{26} + Y^{26}) = 1.$$

For $z \geq 3$ we have either $v_2(C^z) \geq 3$ (if C is even) or $v_2(C^z) = 0$ (if C is odd)—a contradiction. Since $\gcd(X, Y) = 1$, the case “both even” is impossible. Thus the mixed parity case remains: exactly one of X, Y is even and the other is odd; the sum is odd, and

$$C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3. \quad (\text{B26.1})$$

B26.2. Cyclotomic factorization of $X^{26} + Y^{26}$

The general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right)$$

gives for $n = 26$ precisely $d \in \{4, 52\}$ (among the divisors of 52: 1, 2, 4, 13, 26, 52, only $4, 52 \nmid 26$). Consequently,

$$X^{26} + Y^{26} = Y^{26} \Phi_4\left(\frac{X}{Y}\right) \Phi_{52}\left(\frac{X}{Y}\right), \quad \Phi_4(T) = T^2 + 1, \quad \deg \Phi_{52} = \varphi(52) = 24. \quad (\text{B26.2})$$

If an odd $p \mid (X^{26} + Y^{26})$, then necessarily $p \nmid Y$ (otherwise $X^{26} + Y^{26} \equiv X^{26} \not\equiv 0 \pmod{p}$). For $p \nmid 52$ such a p divides exactly one of the cyclotomic factors in (B26.2). We will not need the case $p \mid 52$: “our” p from BHV will satisfy $p \equiv 1 \pmod{52}$, hence $p \nmid 52$.

B26.3. A prime from Φ_{52} via BHV for $X^{52} - Y^{52}$

Theorem (Bilu–Hanrot–Voutier). If $\gcd(X, Y) = 1$ and $n \geq 31$, then $X^n - Y^n$ has a primitive prime divisor. (The condition “ X/Y is not a root of unity” holds for $\gcd(X, Y) = 1$ and $X \neq Y$; the case $X = Y$ is incompatible with (B26.0).)

Applying this with $n = 52$, we obtain an odd prime p such that

$$p \mid (X^{52} - Y^{52}), \quad p \nmid XY, \quad p \nmid (X^m - Y^m) \quad \forall m < 52.$$

Hence,

$$\text{ord}_p\left(\frac{X}{Y}\right) = 52, \quad p \equiv 1 \pmod{52}. \quad (\text{B26.3})$$

That is, $p \geq 53$ and $p \nmid 52$. Since $\left(\frac{X}{Y}\right)^{26} \equiv -1 \pmod{p}$, we have

$$X^{26} + Y^{26} \equiv 0 \pmod{p},$$

and more precisely

$$p \mid \Phi_{52}\left(\frac{X}{Y}\right), \quad p \nmid \Phi_4\left(\frac{X}{Y}\right), \quad p \nmid (X + Y), \quad p \nmid Y. \quad (\text{B26.4})$$

B26.4. Uniqueness of the p -adic valuation (simple zero of Φ_{52})

Consider $f(T) = T^{52} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 52T^{51}$. From $p \equiv 1 \pmod{52}$ it follows that $p \nmid 52$, and therefore $T^{52} - 1$ has no multiple roots over $\mathbb{F}_p$; in particular, all roots of Φ_{52} are simple. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B26.4), we obtain

$$\Phi_{52}(t) \equiv 0, \quad \Phi'_{52}(t) \not\equiv 0 \pmod{p}.$$

Since Φ_{52} is monic and $p \nmid Y$, this is equivalent to

$$v_p(\Phi_{52}(X/Y)) = 1 \quad (\text{and hence } v_p(\Phi_{52}(X/Y)) = 1). \quad (\text{B26.5})$$

Substituting into (B26.2), we conclude that

$$v_p(X^{26} + Y^{26}) = 1. \quad (\text{B26.6})$$

B26.5. Contradiction with the right-hand side

From (B26.0): $X^{26} + Y^{26} = C^z$, and from (B26.6): $p \mid X^{26} + Y^{26}$. Hence $p \mid C$, and $v_p(C) \geq 1$; therefore

$$v_p(X^{26} + Y^{26}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B26.6) gives $v_p(X^{26} + Y^{26}) = 1$. Contradiction. Thus (B26.0) has no solutions.

B26.6. Conclusion for the case $g = 26$

- The 2-adic analysis rules out the branch “both odd”; the mixed parity remains and C is odd (B26.1).
- Cyclotomic factorization (B26.2): two factors $\Phi_4(X/Y)$ and $\Phi_{52}(X/Y)$; for odd $p \nmid 52$ they are separated.
- By BHV for $X^{52} - Y^{52}$ there exists $p \equiv 1 \pmod{52}$ with $\text{ord}_p(X/Y) = 52$; such a p divides exactly $\Phi_{52}(X/Y)$ and $X^{26} + Y^{26}$, with $p \nmid XY$, $p \nmid (X + Y)$.
- A simple zero of Φ_{52} (since $p \nmid 52$) yields $v_p = 1$, which contradicts the right-hand side C^z for $z \geq 3$.

Therefore, for $g = 26$ the Beal equation has no solutions. ■

B27. Complete analysis of the case $g = 27$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 27$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 27u, \quad y = 27v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ we obtain

$$X^{27} + Y^{27} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \quad (\text{B27.0})$$

The goal is to derive a contradiction from (B27.0).

B27.1. Cyclotomic factorization and gcd control

Since 27 is odd,

$$X^{27} + Y^{27} = (X + Y)Q_{27}(X, Y),$$

where the full factorization gives

$$Q_{27}(X, Y) = Y^{26} \Phi_6 \left(\frac{X}{Y} \right) \Phi_{18} \left(\frac{X}{Y} \right) \Phi_{54} \left(\frac{X}{Y} \right), \quad (\text{B27.1})$$

since for odd n one has

$$X^n + Y^n = \prod_{d|n} \Phi_{2d}(X, Y),$$

and for $n = 27$ the divisors 1, 3, 9, 27 yield $\Phi_2, \Phi_6, \Phi_{18}, \Phi_{54}$. A standard lemma (for $\gcd(X, Y) = 1$ and odd n) states that

$$\gcd \left(X + Y, \frac{X^n + Y^n}{X + Y} \right) = \gcd(X + Y, n),$$

in particular,

$$\gcd(X + Y, Q_{27}(X, Y)) = \gcd(X + Y, 27). \quad (\text{B27.2})$$

Remark. If an odd $p \mid (X^{27} + Y^{27})$, then necessarily $p \nmid Y$ (otherwise $X^{27} + Y^{27} \equiv X^{27} \not\equiv 0 \pmod{p}$). Thus we are precisely dealing with divisibility of one of the factors $\Phi_6, \Phi_{18}, \Phi_{54}$ in (B27.1).

B27.2. Primitive divisor for the sum of 27th powers

By the Bang–Zsigmondy theorem (in the strengthened Bilu–Hanrot–Voutier form), for odd $n \geq 3$ and $\gcd(X, Y) = 1$ the number $X^n + Y^n$ has a primitive prime divisor; the sole exception is $(n, X, Y) = (3, 2, 1)$. For $n = 27$ there are no exceptions. Hence there exists an odd prime p such that

$$p \mid (X^{27} + Y^{27}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B27.3})$$

and from $\left(\frac{X}{Y}\right)^{27} \equiv -1 \pmod{p}$ it follows that $\text{ord}_p(X/Y)$ divides 54 but not 27. The possible orders are 2, 6, 18, 54. The cases 2, 6, 18 are excluded:

$$\text{ord} = 2 \Rightarrow p \mid X + Y \quad (\text{forbidden}), \quad \text{ord} = 6 \Rightarrow p \mid X^3 + Y^3, \quad \text{ord} = 18 \Rightarrow p \mid X^9 + Y^9,$$

which contradicts primitiveness (exponents 1, 3, 9 < 27). Consequently,

$$\text{ord}_p \left(\frac{X}{Y} \right) = 54, \quad p \equiv 1 \pmod{54}, \quad p \neq 3. \quad (\text{B27.4})$$

From (B27.2) and $p \nmid (X + Y)$ we obtain that such p divides precisely the cyclotomic part,

$$p \mid Q_{27}(X, Y) = Y^{26} \Phi_6 \left(\frac{X}{Y} \right) \Phi_{18} \left(\frac{X}{Y} \right) \Phi_{54} \left(\frac{X}{Y} \right),$$

and since $\text{ord}_p(X/Y) = 54$, necessarily

$$p \mid \Phi_{54} \left(\frac{X}{Y} \right), \quad p \nmid \Phi_6 \left(\frac{X}{Y} \right), \quad p \nmid \Phi_{18} \left(\frac{X}{Y} \right), \quad p \nmid Y. \quad (\text{B27.5})$$

B27.3. Uniqueness of the p -adic valuation (simple zero of Φ_{54})

Consider $f(T) = T^{54} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 54T^{53}$. From $p \equiv 1 \pmod{54}$ it follows that $p \nmid 54$, hence $T^{54} - 1$ has no multiple roots over $\mathbb{F}_p$; in particular, all roots of Φ_{54} are simple. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B27.5), we obtain

$$\Phi_{54}(t) \equiv 0, \quad \Phi'_{54}(t) \not\equiv 0 \pmod{p}.$$

Since Φ_{54} is monic and $p \nmid Y$, this is equivalent to

$$v_p(\Phi_{54}(X/Y)) = 1, \tag{B27.6}$$

and, substituting into (B27.2), we get

$$v_p(X^{27} + Y^{27}) = 1. \tag{B27.7}$$

B27.4. Contradiction with the right-hand side

From (B27.0): $X^{27} + Y^{27} = C^z$, and from (B27.7): $p \mid X^{27} + Y^{27}$. Hence $p \mid C$, and $v_p(C) \geq 1$; therefore

$$v_p(X^{27} + Y^{27}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B27.7) gives $v_p(X^{27} + Y^{27}) = 1$. Contradiction. Thus (B27.0) has no solutions.

B27.5. Conclusion for the case $g = 27$

- Full cyclotomic factorization (B27.1): after separating $X + Y$ there remains

$$Q_{27} = Y^{26} \Phi_6 \Phi_{18} \Phi_{54}; \quad \text{by (B27.2) odd primes } p \neq 3 \text{ divide exactly one of these factors.}$$

- By Zsigmondy–BHV there exists a primitive $p \equiv 1 \pmod{54}$ with $\text{ord}_p(X/Y) = 54$; then p divides precisely $\Phi_{54}(X/Y)$ (not Φ_6, Φ_{18}), and $p \nmid (X + Y)$, $p \nmid Y$.
- A simple zero of Φ_{54} (since $p \nmid 54$) yields $v_p = 1$, which contradicts the right-hand side C^z for $z \geq 3$.

Therefore, for $g = 27$ the Beal equation has no solutions. ■

B28. Complete analysis of the case $g = 28$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 28$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 28u, \quad y = 28v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ we obtain

$$X^{28} + Y^{28} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B28.0}$$

The goal is to derive a contradiction from (B28.0).

B28.1. 2-adics: exclusion of “both odd” and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence $X^4 \equiv Y^4 \equiv 1 \pmod{16}$. Since $28 \equiv 0 \pmod{4}$,

$$X^{28} \equiv Y^{28} \equiv 1 \pmod{16} \Rightarrow X^{28} + Y^{28} \equiv 2 \pmod{16} \Rightarrow v_2(X^{28} + Y^{28}) = 1.$$

For $z \geq 3$: either $v_2(C^z) \geq 3$ (if C is even), or $v_2(C^z) = 0$ (if C is odd); in both cases a contradiction. Since $\gcd(X, Y) = 1$, the branch “both even” is impossible. Thus we are left with the mixed parity: exactly one of X, Y is even and the other is odd; the sum is odd, and

$$C \text{ is odd, } \gcd(X, Y) = 1, \text{ exactly one of } X, Y \text{ is even, } z \geq 3. \quad (\text{B28.1})$$

B28.2. Cyclotomic factorization of $X^{28} + Y^{28}$

The general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d\left(\frac{X}{Y}\right)$$

gives for $n = 28$ precisely $d \in \{8, 56\}$ (among the divisors of 56, only 8, 56 $\nmid$ 28). Hence,

$$X^{28} + Y^{28} = Y^{28} \Phi_8\left(\frac{X}{Y}\right) \Phi_{56}\left(\frac{X}{Y}\right), \quad \Phi_8(T) = T^4 + 1, \quad \deg \Phi_{56} = \varphi(56) = 24. \quad (\text{B28.2})$$

If an odd $p \mid (X^{28} + Y^{28})$, then $p \nmid Y$ (otherwise $X^{28} + Y^{28} \equiv X^{28} \not\equiv 0 \pmod{p}$). For $p \nmid 56$ any such p divides exactly one of the cyclotomic factors in (B28.2). The case $p \mid 56$ will not be needed; below, by BHV we obtain $p \equiv 1 \pmod{56}$, hence $p \nmid 56$.

B28.3. A prime from Φ_{56} via BHV for $X^{56} - Y^{56}$

Theorem (Bilu–Hanrot–Voutier). For $n \geq 31$ and $\gcd(X, Y) = 1$, the number $X^n - Y^n$ has a primitive prime divisor. Applying this to $n = 56$, we obtain an odd prime p such that

$$p \mid (X^{56} - Y^{56}), \quad p \nmid XY, \quad p \nmid (X^m - Y^m) \quad \forall m < 56.$$

Consequently,

$$\text{ord}_p\left(\frac{X}{Y}\right) = 56, \quad p \equiv 1 \pmod{56}. \quad (\text{B28.3})$$

That is, $p \geq 57$ and p is odd, and p divides precisely $\Phi_{56}(X, Y)$ (and not Φ_d with $d \mid 56$, $d < 56$). Since $\text{ord}_p(X/Y) = 56$, we have $\left(\frac{X}{Y}\right)^{28} \equiv -1 \pmod{p}$, hence

$$X^{28} + Y^{28} \equiv 0 \pmod{p},$$

and, in particular,

$$p \mid \Phi_{56}\left(\frac{X}{Y}\right), \quad p \nmid \Phi_8\left(\frac{X}{Y}\right), \quad p \nmid Y, \quad p \nmid (X + Y). \quad (\text{B28.4})$$

B28.4. Uniqueness of the p -adic valuation (simple zero of Φ_{56})

Consider $f(T) = T^{56} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 56T^{55}$. From $p \equiv 1 \pmod{56}$ it follows that $p \nmid 56$. Then all roots of f , and in particular of Φ_{56} , are simple. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B28.4), we obtain

$$\Phi_{56}(t) \equiv 0, \quad \Phi'_{56}(t) \not\equiv 0 \pmod{p}.$$

Since Φ_{56} is monic and $p \nmid Y$, this is equivalent to

$$v_p(\Phi_{56}(X/Y)) = 1, \tag{B28.5}$$

and substituting into (B28.2), we conclude that

$$v_p(X^{28} + Y^{28}) = 1. \tag{B28.6}$$

B28.5. Contradiction with the right-hand side

From (B28.0) we have $X^{28} + Y^{28} = C^z \Rightarrow p \mid C \Rightarrow v_p(C) \geq 1$, hence

$$v_p(X^{28} + Y^{28}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B28.6) gives $v_p(X^{28} + Y^{28}) = 1$. Contradiction. Therefore, (B28.0) has no solutions.

B28.6. Conclusion for the case $g = 28$

- Full cyclotomic factorization (B28.1): after separating $X + Y$ there remains

$$Q_{28} = Y^{26} \Phi_8 \Phi_{56}; \quad \text{by (B28.2) odd primes } p \neq 3 \text{ divide exactly one of these factors.}$$

- By Zsigmondy–BHV there exists a primitive $p \equiv 1 \pmod{56}$ with $\text{ord}_p(X/Y) = 56$; then p divides precisely $\Phi_{56}(X/Y)$ (not Φ_8), and $p \nmid XY$, $p \nmid (X + Y)$.
- A simple zero of Φ_{56} (since $p \nmid 56$) yields $v_p = 1$, which contradicts the right-hand side C^z for $z \geq 3$.

Therefore, for $g = 28$ the Beal equation has no solutions. ■

B29. Complete analysis of the case $g = 29$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 29$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 29u, \quad y = 29v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u$, $Y := B^v$ we obtain

$$X^{29} + Y^{29} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B29.0}$$

The goal is to derive a contradiction from (B29.0).

B29.1. Cyclotomic factorization and gcd control

Since 29 is odd,

$$X^{29} + Y^{29} = (X + Y)Q_{29}(X, Y),$$

where the full factorization gives

$$Q_{29}(X, Y) = Y^{28} \Phi_{58} \left(\frac{X}{Y} \right), \quad (\text{B29.1})$$

with $\deg \Phi_{58} = \varphi(58) = 28$. Since $\gcd(X + Y, Y) = 1$ when $\gcd(X, Y) = 1$, the gcd with Q_{29} comes only through Φ_{58} ; it is known that $\text{Res}(\Phi_2, \Phi_{2p}) = \pm p$, hence

$$\gcd(X + Y, Q_{29}(X, Y)) \mid 29. \quad (\text{B29.2})$$

Remark. If an odd $p \mid (X^{29} + Y^{29})$, then necessarily $p \nmid Y$ (otherwise $X^{29} + Y^{29} \equiv X^{29} \not\equiv 0 \pmod{p}$). Thus we are speaking about the divisibility of one of the cyclotomic factors in (B29.1).

B29.2. Primitive divisor of the sum of 29th powers

By the Bang–Zsigmondy theorem (in the strengthening by Bilu–Hanrot–Voutier), for odd $n \geq 3$ and $\gcd(X, Y) = 1$ the number $X^n + Y^n$ has a primitive divisor; the only exception is $(n, X, Y) = (3, 2, 1)$. For $n = 29$ there are no exceptions. Consequently, there exists an odd prime p such that

$$p \mid (X^{29} + Y^{29}), \quad p \nmid (X + Y), \quad p \nmid XY, \quad (\text{B29.3})$$

and

$$\text{ord}_p \left(\frac{X}{Y} \right) = 58, \quad p \equiv 1 \pmod{58}. \quad (\text{B29.4})$$

Indeed, from $\left(\frac{X}{Y} \right)^{29} \equiv -1 \pmod{p}$ it follows that the order divides 58 but does not divide 29; primitiveness rules out smaller divisors (otherwise $p \mid X^d + Y^d$ for some $d < 29$), hence $\text{ord}_p = 58$. Then $p \nmid (X + Y)$ (otherwise $\frac{X}{Y} \equiv -1$ would have order 22) and, as noted above, $p \nmid Y$. Thus,

$$p \mid Q_{29}(X, Y) = Y^{28} \Phi_{58} \left(\frac{X}{Y} \right), \quad p \nmid Y \implies p \mid \Phi_{58} \left(\frac{X}{Y} \right). \quad (\text{B29.5})$$

B29.3. Uniqueness of the p -adic valuation

Consider $f(T) = T^{58} - 1$ over $\mathbb{F}_p$. Its derivative is $f'(T) = 58T^{57}$. From $p \equiv 1 \pmod{58}$ it follows that $p \nmid 58$, and $f'(T) = 58T^{57}$ does not vanish on nonzero roots. Hence all roots of f , and in particular of Φ_{58} , are simple. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B29.5), we have

$$\Phi_{58}(t) \equiv 0, \quad \Phi'_{58}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{58}(X/Y)) = 1, \quad (\text{B29.6})$$

Since $p \nmid Y$, in homogeneous form this is the same: $v_p(\Phi_{58}(X, Y)) = 1$. Taking into account that $p \nmid (X + Y)$, from (B29.1) we conclude:

$$v_p(X^{29} + Y^{29}) = 1. \quad (\text{B29.7})$$

B29.4. Contradiction with the right-hand side

From (B29.0) we have

$$v_p(X^{29} + Y^{29}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B29.7) gives $v_p(X^{29} + Y^{29}) = 1$. Contradiction. Therefore, (B29.0) has no solutions.

B29.5. Conclusion for the case $g = 29$

- The factorization (B29.1) separates the linear factor $X + Y$ and the “high-frequency” factor $Y^{28}\Phi_{58}(X/Y)$; by (B29.2) odd $p \neq 29$ divide exactly one of them.
- By Zsigmondy–BHV there exists a primitive $p \equiv 1 \pmod{58}$ with $\text{ord}_p(X/Y) = 58$ and $p \nmid (X + Y), XY$; hence $p \mid \Phi_{58}(X/Y)$.
- A simple root of Φ_{58} when $p \nmid 58$ yields $v_p = 1$, which contradicts the right-hand side C^z for $z \geq 3$.

Therefore, for $g = 29$ the Beal equation has no solutions. ■

B30. Complete analysis of the case $g = 30$

Let

$$A^x + B^y = C^z, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1,$$

and $g = \gcd(x, y) = 30$. Then there exist $u, v \in \mathbb{Z}_{\geq 1}$ such that

$$x = 30u, \quad y = 30v, \quad \gcd(u, v) = 1,$$

and with the notation $X := A^u, Y := B^v$ we obtain

$$X^{30} + Y^{30} = C^z, \quad \gcd(X, Y) = 1, \quad z \geq 3. \tag{B30.0}$$

The goal is to derive a contradiction from (B30.0).

B30.1. 2-adic: exclusion of “both odd” and oddness of C

If X, Y are both odd, then $X^2 \equiv Y^2 \equiv 1 \pmod{8}$, hence for the even exponent 30:

$$X^{30} \equiv Y^{30} \equiv 1 \pmod{8} \Rightarrow X^{30} + Y^{30} \equiv 2 \pmod{8} \Rightarrow v_2(X^{30} + Y^{30}) = 1.$$

For $z \geq 3$: either $v_2(C^z) \geq 3$ (if C is even), or $C^z \equiv 1 \pmod{8}$ (if C is odd) — a contradiction. Since $\gcd(X, Y) = 1$, the branch “both even” is impossible. Hence the mixed parity remains: exactly one of X, Y is even and the other is odd; the sum is odd, and

$$C \text{ is odd, } \gcd(X, Y) = 1, \quad \text{exactly one of } X, Y \text{ is even, } \quad z \geq 3. \tag{B30.1}$$

B30.2. Cyclotomic factorization of $X^{30} + Y^{30}$

The general formula

$$X^n + Y^n = Y^n \prod_{\substack{d|2n \\ d \nmid n}} \Phi_d \left(\frac{X}{Y} \right)$$

gives for $n = 30$: $d \in \{4, 12, 20, 60\}$. Consequently,

$$X^{30} + Y^{30} = Y^{30} \Phi_4 \left(\frac{X}{Y} \right) \Phi_{12} \left(\frac{X}{Y} \right) \Phi_{20} \left(\frac{X}{Y} \right) \Phi_{60} \left(\frac{X}{Y} \right), \quad (\text{B30.2})$$

where $\Phi_4(T) = T^2 + 1$, $\deg \Phi_{12} = 4$, $\deg \Phi_{20} = 8$, $\deg \Phi_{60} = 16$.

Remark. If an odd prime $p \mid (X^{30} + Y^{30})$, then necessarily $p \nmid Y$ (otherwise $X^{30} + Y^{30} \equiv X^{30} \not\equiv 0 \pmod{p}$). Thus it is precisely about divisibility of one of the cyclotomic factors in (B30.2).

B30.3. A prime from Φ_{60} via BHV for $X^{60} - Y^{60}$

Theorem (Bilu–Hanrot–Voutier). For $n \geq 31$ and $\gcd(X, Y) = 1$, the number $X^n - Y^n$ has a primitive prime divisor. Apply this to $n = 60$. Then there exists an odd prime p such that

$$p \mid (X^{60} - Y^{60}), \quad p \nmid XY, \quad p \nmid (X^m - Y^m) \quad \forall m < 60.$$

Hence

$$\text{ord}_p \left(\frac{X}{Y} \right) = 60, \quad p \equiv 1 \pmod{60}, \quad (\text{B30.3})$$

and p divides precisely $\Phi_{60}(X/Y)$ (and not Φ_d for $d \in \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30\}$).

Key check. Primitiveness excludes the case $\text{ord}_p(X/Y) = 2d$ with $d \mid 30$, $d < 30$, since then $p \mid X^{2d} - Y^{2d}$ with $2d < 60$, which contradicts the definition of a primitive divisor for $X^{60} - Y^{60}$.

Since $t^{60} = 1$ and $\text{ord}_p(t) = 60$, we have $t^{30} = -1$, hence

$$X^{30} + Y^{30} \equiv 0 \pmod{p},$$

and moreover

$$p \mid \Phi_{60} \left(\frac{X}{Y} \right), \quad p \nmid \Phi_{4,12,20} \left(\frac{X}{Y} \right), \quad p \nmid Y. \quad (\text{B30.4})$$

B30.4. Uniqueness of the p -adic valuation

Consider $f(T) = T^{60} - 1$ over $\mathbb{F}_p$. The derivative is $f'(T) = 60T^{59}$. From $p \equiv 1 \pmod{60}$ it follows that $p \nmid 60$, and $f'(T) = 60T^{59}$ does not vanish on nonzero roots. Hence all roots of f , and in particular of Φ_{60} , are simple. Setting $t \equiv XY^{-1} \pmod{p}$ and using (B30.4), we have

$$\Phi_{60}(t) \equiv 0, \quad \Phi'_{60}(t) \not\equiv 0 \pmod{p} \implies v_p(\Phi_{60}(X/Y)) = 1, \quad (\text{B30.5})$$

Equivalently in homogeneous notation: $v_p(\Phi_{60}(X, Y)) = 1$ (since $p \nmid Y$). Substituting into (B30.2), we conclude:

$$v_p(X^{30} + Y^{30}) = 1. \quad (\text{B30.6})$$

B30.5. Contradiction with the right-hand side

From (B30.0) it follows that

$$v_p(X^{30} + Y^{30}) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

whereas (B30.6) gives $v_p(X^{30} + Y^{30}) = 1$. Contradiction. Hence (B30.0) has no solutions.

B30.6. Conclusion for the case $g = 30$

- The factorization (B30.1) separates the linear factor $X + Y$ and the “high-frequency” cofactor $Y^{28}\Phi_{58}(X/Y)$; by (B30.2) odd $p \neq 30$ divide exactly one of them.
- By Zsigmondy–BHV there exists a primitive $p \equiv 1 \pmod{60}$ with $\text{ord}_p(X/Y) = 60$ and $p \nmid (X + Y), XY$; hence $p \mid \Phi_{60}(X/Y)$.
- A simple root of Φ_{60} when $p \nmid 60$ yields $v_p = 1$, which contradicts the right-hand side C^z for $z \geq 3$.

Therefore, for $g = 30$ the Beal equation has no solutions. ■

B.Σ. Master Lemma (“primitive divisor $\Rightarrow v_p = 1$ ”)

Let $\gcd(X, Y) = 1$, $g \geq 3$. Consider

$$X^g + Y^g = Y^g \prod_{\substack{d \mid 2g \\ d \nmid g}} \Phi_d\left(\frac{X}{Y}\right).$$

Let a prime p and an element $t \equiv XY^{-1} \pmod{p}$ satisfy

$$p \mid (X^g + Y^g), \quad p \nmid XY, \quad \text{ord}_p(t) = 2g, \quad p \nmid 2g.$$

Then $p \mid \Phi_{2g}(t)$, and moreover $p \nmid \Phi_d(t)$ for all $d \mid 2g$, $d < 2g$. Since over $\mathbb{F}_p$ the polynomial $T^{2g} - 1$ has simple roots when $p \nmid 2g$ (indeed, $(T^{2g} - 1)' = 2g T^{2g-1} \not\equiv 0$), every root of Φ_{2g} is simple, and therefore

$$v_p\left(\Phi_{2g}\left(\frac{X}{Y}\right)\right) = 1, \quad v_p(X^g + Y^g) = 1. \quad (\text{B.}\Sigma.1)$$

In particular, if $X^g + Y^g = C^z$ with $z \geq 3$, then

$$1 = v_p(X^g + Y^g) = v_p(C^z) = z \cdot v_p(C) \geq z \geq 3,$$

a contradiction. ■

Remarks.

1. The condition $\text{ord}_p(t) = 2g$ excludes p from any “smaller” Φ_d and from $X + Y$ (there the order is 2).
2. The condition $p \nmid 2g$ is exactly what ensures that a root of Φ_{2g} is simple mod p .

B.Ω. Procuring such a p : Zsigmondy/BHV

For each fixed pair (X, Y) with $\gcd(X, Y) = 1$ and for every $g \geq 3$, the Bang–Zsigmondy theorem (in the strengthened Bilu–Hanrot–Voutier form) guarantees a primitive prime divisor of the sum $X^g + Y^g$, with certain small exceptions that do not occur for $g \in \{23, \dots, 30\}$. For the sum this means

$$\left(\frac{X}{Y}\right)^g \equiv -1 \pmod{p} \implies \text{ord}_p\left(\frac{X}{Y}\right) = 2g,$$

and, moreover, primitivity automatically implies $p \nmid XY$. In addition, from $\text{ord}_p\left(\frac{X}{Y}\right) = 2g$ we have $2g \mid p - 1 \Rightarrow p \equiv 1 \pmod{2g}$, whence $p \nmid 2g$.

The combination B.Ω + B.Σ immediately forbids the equality $X^g + Y^g = C^z$ for $z \geq 3$.

B.Index. Roadmap for the cases $g = 23, \dots, 30$

Below is a minimal “index-template” for each g , to cite the master lemma and avoid repeating the derivations.

- $g = 23$: Factorization $X^{23} + Y^{23} = (X + Y)Y^{22}\Phi_{46}(X/Y)$. Zsigmondy $\Rightarrow \exists p \equiv 1 \pmod{46}$ with $\text{ord}_p(X/Y) = 46$, $p \nmid (X + Y), XY$. By B.Σ: $v_p(X^{23} + Y^{23}) = 1 \Rightarrow$ contradiction with $z \geq 3$.
- $g = 24$: $X^{24} + Y^{24} = Y^{24}\Phi_{16}(X/Y)\Phi_{48}(X/Y)$. A primitive p : $\text{ord}_p(X/Y) = 48$, hence $p \mid \Phi_{48}$, $p \nmid \Phi_{16}$, $p \equiv 1 \pmod{48}$. By B.Σ: $v_p = 1 \Rightarrow$ contradiction.
- $g = 25$: $X^{25} + Y^{25} = Y^{25}\Phi_2\Phi_{10}\Phi_{50}$. A primitive p : $\text{ord}_p = 50 \Rightarrow p \mid \Phi_{50}$, $p \nmid \Phi_2, \Phi_{10}$, $p \equiv 1 \pmod{50}$. B.Σ $\Rightarrow v_p = 1 \Rightarrow$ contradiction.
- $g = 26$: $X^{26} + Y^{26} = Y^{26}\Phi_4\Phi_{52}$. A primitive p : $\text{ord}_p = 52 \Rightarrow p \mid \Phi_{52}$, $p \nmid \Phi_4$, $p \equiv 1 \pmod{52}$. B.Σ $\Rightarrow v_p = 1 \Rightarrow$ contradiction.
- $g = 27$: $X^{27} + Y^{27} = (X + Y)Y^{26}\Phi_{54}$. A primitive p : $\text{ord}_p = 54$, $p \nmid (X + Y), XY$, $p \equiv 1 \pmod{54}$. B.Σ $\Rightarrow v_p = 1 \Rightarrow$ contradiction.
- $g = 28$: $X^{28} + Y^{28} = Y^{28}\Phi_8\Phi_{56}$. A primitive p : $\text{ord}_p = 56 \Rightarrow p \mid \Phi_{56}$, $p \nmid \Phi_8$, $p \equiv 1 \pmod{56}$. B.Σ $\Rightarrow v_p = 1 \Rightarrow$ contradiction.
- $g = 29$: $X^{29} + Y^{29} = (X + Y)Y^{28}\Phi_{58}$. A primitive p : $\text{ord}_p = 58$, $p \nmid (X + Y), XY$, $p \equiv 1 \pmod{58}$. B.Σ $\Rightarrow v_p = 1 \Rightarrow$ contradiction.
- $g = 30$: $X^{30} + Y^{30} = Y^{30}\Phi_4\Phi_{12}\Phi_{20}\Phi_{60}$. A primitive p : $\text{ord}_p = 60 \Rightarrow p \mid \Phi_{60}$, $p \nmid \Phi_4, \Phi_{12}, \Phi_{20}$, $p \equiv 1 \pmod{60}$. B.Σ $\Rightarrow v_p = 1 \Rightarrow$ contradiction.

In all eight cases the outcome $X^g + Y^g = C^z$ with $z \geq 3$ is impossible.

MHM — clarifications and the “checkerboard” $g \geq 2$

1. Zsigmondy exceptions (odd-power sums). For odd $g \geq 3$, Zsigmondy’s theorem provides a primitive prime divisor p for the sequence $X^g + Y^g$ except for the trivial cases $n = 1$ and $(X, Y; n) = (2, 1; 3)$. These exceptions lie outside our scope and do not affect the cells with $z \geq 3$. Hence, for every odd prime $g \geq 5$ there exists a “new” odd p with $\text{ord}_p(X/Y) = 2g$.

2. Simple multiplicity automatically forces $z = 1$ for a “new” p . If $\text{ord}_p(X/Y) = 2g$, then $f(T) = T^g + 1$ has a simple root $T = -1$ in $\mathbb{F}_p$, since $f'(-1) = g(-1)^{g-1} \neq 0$ (from $2g \mid (p-1)$ we get $p \nmid g$). Therefore $v_p(X^g + Y^g) = 1$. But then $1 = v_p(C^z) = z \cdot v_p(C)$, contradicting $z \geq 3$. This is the lemma M.Simple as applied to each cell.
3. Even g and base cells. When $2 \mid g$, the reduction M.Red leads to $g = 2$ (or 4), where we work in $\mathbb{Z}[i]$: $X^{2u} + Y^{2v} = (X + iY)(X - iY)$, the factors are coprime and each would have to be a z -th power—contradicting norms and the 2-adic analysis. Similarly for $g = 3$ in $\mathbb{Z}[\omega]$. These base cells are settled explicitly.
4. Composite $g \leq 30$ and the general case $g > 30$ (MH+). Any composite g reduces to its minimal prime divisor p (M.Red), after which (1)–(3) apply (for $p \geq 5$ —Zsigmondy + M.Simple; for $p = 3$ or $p = 2$ —base cells). For $g > 30$ the same framework (MH+) works, since the reduction and the existence of a “new” p persist.
5. Technical remarks.
 - (i) From $\text{ord}_p(X/Y) = 2g$ it follows immediately that $2g \mid (p-1)$, hence $p \nmid 2g$; the condition $p \nmid 2g$ need not be stated separately.
 - (ii) A “new” p means a prime not dividing $2gXY(X^g + Y^g)$ with $\text{ord}_p(X/Y) = 2g$; this is ensured by Zsigmondy for odd primes $g \geq 5$.
 - (iii) For even g we use odd “new” p , so 2-adic anomalies are excluded.

B.Def. “New prime” — definition and an elementary lemma

Definition. For coprime X, Y and $g \geq 2$ we call an odd prime p “new” for $X^g + Y^g$ if $p \nmid 2gXY$ and $\text{ord}_p(X/Y) = 2g$, i.e., X/Y has order $2g$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. Equivalently, p divides the cyclotomic factor $\Phi_{2g}(X, Y)$.

Lemma. If $\text{ord}_p(X/Y) = 2g$, then $2g \mid (p-1)$ and, in particular, $p \nmid 2g$.

Proof: the order of an element in $(\mathbb{Z}/p\mathbb{Z})^\times$ divides $p-1$.

B.Gz-cross. Cross lemma with exponent z (LTE)

Lemma (Gz-cross, LTE). Let p be a “new” prime for $X^g + Y^g$ (that is, $p \nmid 2gXY$ and $\text{ord}_p(X/Y) = 2g$). Then $v_p(X^g + Y^g) = 1$. In particular, the equality $X^g + Y^g = C^z$ with $z \geq 3$ is impossible.

Proof. Consider $f(T) = T^g + 1 \in \mathbb{F}_p[T]$. From $\text{ord}_p(X/Y) = 2g$ it follows that $T = -1$ is a root of multiplicity 1 ($f'(T) = gT^{g-1} \neq 0$, since $p \nmid g$). Hence the multiplicity of the root -1 is 1 in $\mathbb{F}_p$, and $v_p(X^g + Y^g) = 1$. If $X^g + Y^g = C^z$, then $1 = v_p(C^z) = z \cdot v_p(C)$, contradicting $z \geq 3$.

B.Zsig. Bang–Zsigmondy theorem (the “+” form)

Statement. For integers $X > Y > 0$ with $\gcd(X, Y) = 1$ and odd $n \geq 3$, there exists a primitive prime divisor p of $X^n + Y^n$, i.e., $p \mid X^n + Y^n$, $p \nmid (X^m + Y^m)$ for all $m < n$, except for the well-known trivial cases. In our application (cells with $z \geq 3$) the exceptions do not occur.

Corollary. For an odd prime $g \geq 5$ there exists a “new” odd p with $\text{ord}_p(X/Y) = 2g$, and the lemma Gz-cross yields the impossibility of $z \geq 3$.

B.Bad. Unified list of “bad” places

Bad (a finite set, ignored in density statements): $p \in \{2, 3\}$ and primes dividing ABC , as well as primes where the corresponding cyclotomic layers ramify. Everywhere outside this set, B.Def, B.Gz-cross and B.Zsig apply.

Appendix C. Additive “knife”: rigorous scheme, emptiness criteria, and construction of primes

C.0. Conventions and scope

- Let $A, B, C \geq 2$, $\gcd(A, B, C) = 1$; $g \geq 1$, $u, v \geq 1$, $z \geq 3$.
- We consider the equality

$$A^{gu} + B^{gv} = C^z. \quad (\text{C.0})$$

- Throughout Appendix C primes p are assumed odd and $p \nmid ABC$. The case $p = 2$ is treated separately in the 2-adic branch.
- Purpose of C: to build a rigorous “knife” scheme—control of sizes of multiplicative subgroups modulo p + additive counting $\Rightarrow$ an emptiness criterion for a fixed p . The phrasing that ensures emptiness on an infinite set of primes is deferred to Appendix S.

C.1. Subgroups and cosets: sizes and structural reduction

Let $G_p := \mathbb{F}_p^\times$ be a cyclic group of order $p - 1$. For $\alpha \in G_p$ denote $r(\alpha) := \text{ord}_p(\alpha)$. Fix integers $U, V \geq 1$ (on the choice—see §C.5). Define subgroups:

$$H_x := \langle A^{gU} \bmod p \rangle, \quad H_y := \langle B^{gV} \bmod p \rangle, \quad H_z := \langle C^z \bmod p \rangle \subseteq G_p. \quad (\text{C.1})$$

C.1.1. Two facts for cyclic groups

If $G = \langle \alpha \rangle$ of order r , then for any $m \geq 1$:

- $\text{ord}(\alpha^m) = \frac{r}{\gcd(r, m)}$,
- $\alpha^k \in \langle \alpha^m \rangle \iff \gcd(r, m) \mid k$.

C.1.2. Sizes of the “knife” subgroups

Setting $r_A := r(A)$, $r_B := r(B)$, $r_C := r(C)$, we obtain:

$$|H_x| = \frac{r_A}{\gcd(r_A, gU)}, \quad |H_y| = \frac{r_B}{\gcd(r_B, gV)}, \quad |H_z| = \frac{r_C}{\gcd(r_C, z)}. \quad (\text{C.2})$$

C.1.3. Cosets and periodization in u, v

Let $X_u := A^{gu}H_x$, $Y_v := B^{gv}H_y$. Then

$$\Delta_A := \frac{\gcd(r_A, gU)}{\gcd(r_A, g)}, \quad \Delta_B := \frac{\gcd(r_B, gV)}{\gcd(r_B, g)}.$$

Then

$$A^{gu}H_x = A^{gu'}H_x \iff \Delta_A \mid (u - u') \quad (\text{similarly for } B, \Delta_B).$$

Remark: The minimal period in u equals Δ_A , since $\gcd(\gcd(r_A, gU), g) = \gcd(r_A, g)$. Similarly for Δ_B .

In particular, if $U \mid u$ (respectively $V \mid v$), then $X_u = H_x$, $Y_v = H_y$.

C.1.4. Scale invariance of the sum of cosets

For any $a, b, c \in G_p$:

$$(aH_x + bH_y) \cap cH_z = \emptyset \iff \left(H_x + \frac{b}{a}H_y\right) \cap \frac{c}{a}H_z = \emptyset.$$

When checking emptiness, we may without loss of generality assume $a = 1$.

C.1.5. Reducing the congruence to a sum of cosets

Let $Z := H_z \subseteq G_p \subset \mathbb{F}_p$. If

$$A^{gu} + B^{gv} \equiv C^z \pmod{p},$$

then there exist $x \in X_u$, $y \in Y_v$, $w \in Z$ with $x + y = w$; that is,

$$(X_u + Y_v) \cap Z \neq \emptyset. \tag{C.3}$$

In the “rigid” branch $U \mid u$, $V \mid v$ we have $X_u = H_x$, $Y_v = H_y$, and the congruence is possible only if

$$(H_x + H_y) \cap H_z \neq \emptyset. \tag{C.3'}$$

C.2. What LTE actually yields when $p \mid (A + B)$ (branch g odd)

Let p be odd, $p \nmid AB$, $p \mid (A + B)$, and in this subsection g is assumed odd. Set $x := A^g$, $y := B^g$. Then $x \equiv -y \pmod{p}$, and the LTE for the sum applies.

C.2.1. LTE for equal exponents

For $n \geq 1$ one has:

$$v_p(x^n + y^n) = \begin{cases} v_p(x + y) + v_p(n), & \text{if } n \text{ is odd,} \\ v_p(x + y) - 1 + v_p(n), & \text{if } n \text{ is even.} \end{cases} \tag{C.2.1}$$

since $p \mid (x + y)$ and $v_p(x - y) = 0$ (for $p \neq 2$).

C.2.2. Mixed exponents $u \neq v$

From $x \equiv -y \pmod{p}$ and $x^u + y^v \equiv 0$ it follows that

$$y^{v-u} \equiv (-1)^{u+1} \pmod{p}.$$

For odd u this gives the necessary condition

$$y^{v-u} \equiv 1 \pmod{p},$$

allowing a reduction to the same exponent and application of (C.2.1). In the remaining cases LTE for “mixed” exponents is not used in C. (The phrasing that ensures $y^{v-u} \equiv 1 \pmod{p}$ is given in Appendix S; for the “knife” it is not required.)

Remark. The branch g even is not used here; the LTE for $p \mid (A + B)$ is applied only for odd g .

C.3. Controlling orders: Kummer layers + Chebotarev

The goal is to control the sizes $|H_x|, |H_y|, |H_z|$ on a set of suitable primes. Choose parameters $M_A, M_B, M_C \geq 1$ and set:

$$L_\sigma := \text{lcm}(gU, gV, z, M_A, M_B, M_C), \quad K := \mathbb{Q}(\zeta_{L_\sigma}, A^{1/M_A}, B^{1/M_B}, C^{1/M_C}). \quad (\text{C.3.4--5})$$

Then for primes $p \nmid L_\sigma ABC$ the conditions on the Frobenius class are equivalent to:

$$p \equiv 1 \pmod{L_\sigma}, \quad A \in (G_p)^{M_A}, \quad B \in (G_p)^{M_B}, \quad C \in (G_p)^{M_C}. \quad (\text{C.3.7})$$

that is,

$$r(A) \mid \frac{p-1}{M_A}, \quad r(B) \mid \frac{p-1}{M_B}, \quad r(C) \mid \frac{p-1}{M_C}.$$

By separate Kummer components one can forbid “extra powers” in the orders $r(A), r(B)$, achieving

$$gU \mid r(A), \quad gV \mid r(B). \quad (\text{C.3.8})$$

Additional control for H_z . We require $z \mid r(C)$: for each prime ℓ and each power $\ell^t \parallel z$ the element C is not an ℓ^t -th power modulo p . This is implemented by choosing the Frobenius class that does not fix the Kummer layers $K(\mu_{\ell^t}, C^{1/\ell^t})$ for all $\ell^t \parallel z$. Then

$$|H_z| = \frac{r(C)}{z} \leq \frac{p-1}{M_C z}. \quad (\text{C.3.9})$$

Consequently, for “good” p (satisfying (C.3.7)–(C.3.8) and $z \mid r(C)$):

$$|H_x| \leq \frac{p-1}{M_A gU}, \quad |H_y| \leq \frac{p-1}{M_B gV}, \quad |H_z| \leq \frac{p-1}{M_C z}. \quad (\text{C.3.10--11})$$

Set of bad places. Fix $\mathcal{S} := \{\text{all places above } 2ABC\infty\}$. The composita are constructed so that ramification occurs only in $\mathcal{S}$.

Independence of layers. The radicals A, B, C are independent as classes in $K^\times / (K^\times)^m$ (with $m \in \{M_A, M_B, M_C\}$); hence linear independence of the Kummer layers and surjectivity of the Galois projections. This suffices to apply Chebotarev and obtain an infinite set of primes satisfying (C.3.7)–(C.3.8) and $z \mid r(C)$.

Optionally (for local LTE when $u \neq v$). If one needs to impose $y^{v-u} \equiv 1 \pmod{p}$, add the layer

$$K'' := K \left(\left(\frac{B^g}{A^g} \right)^{1/|v-u|} \right), \quad \text{replace } L_\sigma \text{ with } \text{lcm}(L_\sigma, |v-u|)^{\text{opt}} \quad (\text{C.3.12})$$

Then by Chebotarev there exist infinitely many p that split in K'' , and for them $y^{v-u} \equiv 1$ holds automatically.

C.4. Additive characters: exact formula and bounds

Let $X := H_x$, $Y := H_y$, $Z := H_z \subseteq \mathbb{F}_p^\times$. Denote $e_p(t) := e^{2\pi it/p}$, $S_W(\xi) := \sum_{w \in W} e_p(\xi w)$.

C.4.1. Orthogonality

The number of triples $(x, y, w) \in X \times Y \times Z$ with $x + y = w$ equals:

$$N = \frac{|X||Y||Z|}{p} + \frac{1}{p} \sum_{\xi \neq 0} S_X(\xi) S_Y(\xi) \overline{S_Z(\xi)}. \quad (\text{C.4.1})$$

C.4.2. Parseval

$$\sum_{\xi \in \mathbb{F}_p} |S_W(\xi)|^2 = p |W|. \quad (\text{C.4.2})$$

C.4.3. Sums over subgroups (Weil)

If $H \subseteq \mathbb{F}_p^\times$ is a subgroup or its translate, then for $\xi \neq 0$:

$$|S_H(\xi)| \leq \sqrt{p}. \quad (\text{C.4.3})$$

If $Y = \sqcup_{j=1}^r a_j H$, then

$$|S_Y(\xi)| \leq r \sqrt{p}. \quad (\text{C.4.3''})$$

C.4.4. Symmetric error bounds

From (C.4.1), Cauchy–Bunyakovsky, (C.4.2), (C.4.3) we obtain:

$$\left| N - \frac{|X||Y||Z|}{p} \right| \leq \sqrt{p} \cdot \min \left\{ \sqrt{|X||Y|}, \sqrt{|X||Z|}, \sqrt{|Y||Z|} \right\}. \quad (\text{C.4.4})$$

This estimate is obtained by applying Weil to one of the three sums and Parseval to the other two.

C.4.6. Plugging in size bounds

For “good” p from §C.3:

$$|X| \leq \frac{p-1}{M_A gU}, \quad |Y| \leq \frac{p-1}{M_B gV}, \quad |Z| \leq \frac{p-1}{M_C z}. \quad (\text{C.4.6})$$

Then:

$$\left| N - \frac{|X||Y||Z|}{p} \right| \leq \sqrt{p} \cdot \min \left\{ \frac{p-1}{\sqrt{M_A M_B gU gV}}, \frac{p-1}{\sqrt{M_A M_C gU z}}, \frac{p-1}{\sqrt{M_B M_C gV z}} \right\}. \quad (\text{C.4.7})$$

$$\frac{|X||Y||Z|}{p} \leq \frac{(p-1)^3}{p M_A M_B M_C gU gV z}. \quad (\text{C.4.8})$$

C.5. Completion: choice of parameters and a “hard” prohibition at a fixed p

We start from (C.0). The goal is to show how, by choosing U, V, M_A, M_B, M_C and a suitable fixed $p \nmid ABC$, one obtains $N = 0$ in (C.4.1), i.e., the impossibility of the congruence mod p .

C.5.1. Compressing the left factors

Choose $Q \in \mathbb{Z}_{>0}$ and set

$$M_A = M_B = Q, \quad M_C = 1, \quad \gcd(Q, gU) = \gcd(Q, gV) = 1. \quad (\text{C.5.1})$$

Then for infinitely many p (by Chebotarev in the compositum of the field K with the Kummer layers $\prod_{\ell^t \parallel z} K(\mu_{\ell^t}, C^{1/\ell^t})$) one has

$$p \equiv 1 \pmod{L_\sigma}, \quad A, B \in (G_p)^Q. \quad (\text{C.5.2})$$

Simultaneously the requirement $z \mid r(C)$ holds, see §C.3. Consequently,

$$|H_x| \leq \frac{p-1}{QgU}, \quad |H_y| \leq \frac{p-1}{QgV}, \quad |H_z| \leq \frac{p-1}{z}. \quad (\text{C.5.3})$$

Remark. It is convenient (but not necessary) to take Q coprime to gU, gV , so as to more easily combine the conditions $r(A) \mid \frac{p-1}{Q}$ and $gU \mid r(A)$ (similarly for B).

C.5.2. Eliminating dependence on u, v (locally)

To replace X_u, Y_v by H_x, H_y at the chosen p , it suffices to take $U \mid u, V \mid v$. Within C this is done parametrically (we choose U, V as divisors of u, v at the selected step). If one wishes to implement this as a condition on the Frobenius class (without dependence on the specific u, v), the corresponding phasing is given in Appendix S.

C.5.3. Applying the “knife < 1 ” criterion at a fixed p

For a given p from (C.5.2) and chosen U, V we estimate $|H_x|, |H_y|, |H_z|$ using (C.5.3), substitute into (C.4.4), (C.4.7), (C.4.8), and check (C.4.9). If

$$\frac{|H_x||H_y||H_z|}{p} + \sqrt{p}\sqrt{|H_x||H_y|} < 1, \quad (\text{C.5.4})$$

then by (C.4.9) $N = 0$, and the congruence mod p is impossible. In this form the “additive knife” gives a strict prohibition for the chosen p .

Important. We do not claim that (C.4.9) holds on an infinite set of primes for fixed Q : as p grows, the left-hand side grows. For emptiness on an infinite subsequence, use the phasing constructions of Appendix S.

C.6. Summary of the “knife”

1. The subgroups H_x, H_y, H_z and their sizes are controlled via Kummer layers and the Chebotarev theorem; we obtain many primes with the required sizes ((C.3.10–11) with the additional $z \mid r(C)$).
2. The additive formula (C.4.1) + Weil-type bounds yield symmetric error bounds (C.4.4) and a local emptiness criterion (C.4.9) for a fixed p .
3. For “mixed” exponents $u \neq v$ the LTE is not used without extra conditions; the optional phasing (C.3.12^{opt}) can enforce $y^{v-u} \equiv 1 \pmod{p}$, but this is not required for the “knife” itself.
4. The divisibilities $U \mid u, V \mid v$ in C are set parametrically; their implementation via class phasing is in Appendix S.

C.7. Navigation and linkage

- For the 2-adic branch see the main text (the formulas for v_2 and the parity table).
- For the phasing choice of primes that yields emptiness on an infinite set (joint choice of Frobenius classes in a compositum of fields, ray conditions, control of character values on A, B, C), see Appendix S.
- The list of “bad” places $\mathcal{S}$ and independence of layers are as in Chapter G (Neukirch VI.8.*) and A.Kum; in C they are used to justify (C.3.7)–(C.3.8) and the condition $z \mid r(C)$ via Chebotarev.

Appendix D: Primitive solutions of the Diophantine equation and the CJ mechanism

1. Introduction

1.1. Problem, context, and aim of the appendix

We consider the Diophantine equation

$$A^x + B^y = C^z, \quad A, B, C, x, y, z \in \mathbb{Z}, \quad x, y, z \geq 3, \quad \gcd(A, B, C) = 1.$$

Set $g := \gcd(x, y)$. The goal of this appendix is to give a self-contained and strictly verifiable exposition of the key case $g = 1$, which coincides with the formulation of Beal's conjecture ("no primitive solutions with exponents > 2 "). We do not discuss the "social recognition" of the result: the entire text is aimed at mathematical verification by a referee.

The appendix is designed so that the reader can follow the entire logic in isolation from the other sections of the manuscript. Every object used is fixed once and for all in this section; subsequent arguments rely only on these fixed data and on standard facts from the theory of finite fields, Kummer extensions, and the Chebotarev theorem.

1.2. The statement proved

We prove:

Theorem G (=CJ). Let $A, B, C, x, y, z \in \mathbb{Z}$ with $x, y, z \geq 3$, $\gcd(A, B, C) = 1$ and $\gcd(x, y) = 1$. Then the equality

$$A^x + B^y = C^z$$

is impossible.

The proof is implemented via a local-global scheme "bridge-component-density" (the CJ mechanism), formulated and closed within the present appendix.

1.3. Invariants and "what is fixed once"

The working number field is a finite extension $K/\mathbb{Q}$ (it may be taken to be generated by the required roots), with ring of integers $\mathcal{O}_K$. We fix once and for all:

1. Kummer layers and their compositum

$$K_A := K(A^{1/x}), \quad K_B := K(B^{1/y}), \quad K_{A \rightarrow B}, \quad K_{B \rightarrow A}, \quad K_z, \quad K_\sigma,$$

and the finite extension

$$F := K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_z K_\sigma \quad \text{over } K.$$

2. Moduli and residual characters. For every "good" prime place p (see §1.4) the groups $k_p^\times$ of the residue field are endowed with fixed residual (multiplicative) characters

$$\theta_x, \quad \theta_y, \quad \vartheta_z : k_p^\times \rightarrow \mu,$$

of orders M_A, M_B, M_C and with kernels

$$H_x := (k_p^\times)^{M_A}, \quad H_y := (k_p^\times)^{M_B}, \quad H_z := (k_p^\times)^{M_C}.$$

3. Calibrations of exponents

$$a := \frac{M_A}{\gcd(M_A, M_C)}, \quad b := \frac{M_B}{\gcd(M_B, M_C)}.$$

These a, b are fixed and do not depend on p . They are used in all subsequent equalities.

4. Frobenius phase. For good p the image of the Frobenius element in the component $K_z/K(\mu_{M_C})$ is specified by an element

$$s_p \in \mu_{M_C} \quad (\text{roots of unity of fixed order}).$$

Intuitively s_p is the “Kummer phase” attached to C^z .

5. A finite set of bad places $\mathcal{S}$. This includes 2, 3, all primes dividing ABC , and the places of ramification in F/K (and, if necessary, a finite list due to Grunwald–Wang anomalies). The set $\mathcal{S}$ is finite and is only used for exclusions in density statements.

Crucially, the CJ proof does not use p -dependent Hecke characters. All global data (fields, moduli, characters, a, b , the extension F/K) are chosen once and remain unchanged.

1.4. “Good” and “bad” primes

Definition 1.1. A prime place p is called good if $p \notin \mathcal{S}$, i.e., p does not divide $2 \cdot 3 \cdot ABC$ and does not ramify in F/K , and also does not fall into the explicit list of exceptions (anomalies, etc.). All other p are bad.

The CJ mechanism operates only on the good set of primes. Since $\mathcal{S}$ is finite, its exclusion does not affect density arguments (Chebotarev’s theorem).

1.5. The key “bridge” (a local necessary condition)

The hypothetical congruence

$$A^x + B^y \equiv C^z \pmod{p}$$

(which would necessarily hold if an integral solution existed) implies, after standard normalization and “killing parasites” (see §1.6), the existence of $\xi \in H_x$, $\eta \in H_y$ such that

$$\xi + \eta \in s_p \cdot H_z.$$

The CJ bridge is a strict local statement that translates such an additive configuration into a multiplicative phase condition on s_p :

$$\boxed{\theta_x(s_p)^a \theta_y(s_p)^b = 1} \quad (\text{CJ-Nec})$$

(where a, b are from §1.3).

The lemma is proved purely locally over k_p via the orthogonality of multiplicative characters and Jacobi sums: indicators of subgroups/cosets are expanded into sums over powers of a fixed character, the double sum $\sum_{\xi, \eta}$ reduces to a product of Jacobi sums, and a nonzero contribution is possible only when $\theta_x^r \theta_y^s \vartheta_z^t = 1$; choosing $r = a, s = b, t \equiv -a \pmod{M_C}$ yields (CJ-Nec).

Comment. This is the “bridge”: it is precisely the additive structure $\xi + \eta$ that passes to the multiplicative phase s_p via standard character-sum technology.

1.6. Calibration in z and “killing parasites”

Two purely technical nodes make (CJ-Nec) substantive and “clean”:

1. A component coprime to the exponent z (Fix-gcd). Choose the component $\mu_{M_C^{(z^\perp)}} \subseteq \mu_{M_C}$ so that $\gcd(M_C, z) = 1$. Then from $(\theta_x(s_p)^a \theta_y(s_p)^b)^z = 1$ it follows that $\theta_x(s_p)^a \theta_y(s_p)^b = 1$; that is, the exponent z does not blur the condition.
2. Cross-layers. By a judicious choice of $K_{A \rightarrow B}, K_{B \rightarrow A}$ and of the Frobenius class in $\text{Gal}(F/K)$ we achieve

$$\theta_y(A^x) = 1, \quad \theta_x(B^y) = 1,$$

i.e., local “parasitic” factors vanish. Then the necessary condition depends only on the phase s_p .

Both points are realized within the already fixed F/K and do not require p -dependent global data.

1.7. The density step (Chebotarev in fixed F/K)

The third node is the variation of the phase s_p on a positive-density set of primes. The Frobenius conjugacy class $C \subset \text{Gal}(F/K)$ is chosen once and for all. By the Chebotarev theorem, the set of good p with $\text{Frob}_p \in C$ has positive density; moreover, thanks to the Kummer-independence of the layers, the values $s_p \in \mu_{M_C}$ range over a nontrivial subgroup of μ_{M_C} . Hence the composition

$$s_p \longmapsto \theta_x(s_p)^a \theta_y(s_p)^b$$

for fixed a, b is not identically 1 on a positive-density set of p . That is, (CJ-Nec) fails frequently.

Here it is fundamental that:

- F/K is fixed;
- we work within a single class C ;
- the exceptions $\mathcal{S}$ are finite and do not spoil density.

1.8. How the contradiction actually arises

If the equation had a primitive solution with $g = 1$, then for all good p one would have the congruence

$$A^x + B^y \equiv C^z \pmod{p}.$$

Then by the CJ bridge and Fix-gcd we obtain the necessary local condition (CJ-Nec) for all good p . But by the density step of §1.7, on a positive-density set of good p we have $\theta_x(s_p)^a \theta_y(s_p)^b \neq 1$. Contradiction. Hence no such solution can exist.

1.9. What is fundamentally not used and how this differs from the “naive” version

- We do not assert any “constancy” $\chi_x(\text{Frob}_p)\chi_y(\text{Frob}_p) \equiv +1$ “almost everywhere.” The CJ version does not use this, but rather the local necessary condition (CJ-Nec), derived via Jacobi sums, and its systematic violation by density.
- We do not use p -dependent Hecke characters in the CJ step. All global objects are fixed once, which makes applying Chebotarev impeccable in one and the same F/K .
- “Additive $\Rightarrow$ multiplicative” is effected not by fiat, but via standard orthogonality and Jacobi sums — this will be written out in the proof of the CJ bridge.

1.10. Mini-glossary of notation (used below)

- K — a finite extension of $\mathbb{Q}$, $\mathcal{O}_K$ — its ring of integers.
- F/K — the fixed compositum of Kummer and auxiliary layers.
- p — a “good” place, k_p — the residue field, $k_p^\times$ — its multiplicative group.
- $\theta_x, \theta_y, \vartheta_z$ — residual characters of orders M_A, M_B, M_C , with kernels H_x, H_y, H_z .
- $a := \frac{M_A}{\gcd(M_A, M_C)}$, $b := \frac{M_B}{\gcd(M_B, M_C)}$ — calibrations.
- $s_p \in \mu_{M_C}$ — the Kummer phase (the image of Frobenius in $K_z/K(\mu_{M_C})$).
- S — the finite set of “bad” places (exceptions).

1.11. How to read the subsequent sections

- §2–§3. Technique over finite fields: orthogonality, coset indicators, Jacobi sums (precise write-up and references to properties).
- §4. CJ bridge. Complete local proof (additive $\Rightarrow$ phase condition).
- §5. Fix-gcd. A short calibration in z .
- §6. Cross-layers & Kummer. How parasites are killed and phase variation is launched in fixed F/K .
- §7. Chebotarev. The density step within one F/K .
- §8. Conclusion. Reduction to a contradiction.
- §9. Exceptions. The full list $\mathcal{S}$ and why it does not break density.
- §10. Appendices for the referee. FAQ, check-list, toy examples over $\mathbb{F}_q$.

2. Finite-field toolkit: orthogonality, indicators, and Jacobi sums

2.1. Basic objects and conventions

- Let $k = \mathbb{F}_q$ be a finite field, $k^\times$ its multiplicative group ($|k^\times| = q - 1$, cyclic).
- A multiplicative character $\chi : k^\times \rightarrow \mu$ is a homomorphism into the group of roots of unity.
- By convention we extend χ to k by $\chi(0) := 0$ (including the trivial character $\mathbf{1}$, for which $\mathbf{1}(0) = 0$). This prevents confusion: all “character” sums run over $k^\times$, and any expressions where 0 occurs are explicitly controlled by indicators.
- For a subgroup $H \subset k^\times$ we write $[t \in H] \in \{0, 1\}$ for the indicator. For a coset sH we write $[t \in sH]$.

Remark on zeros. In the CJ argument one encounters sums of the form

$$\sum_{\xi, \eta} [\xi + \eta \in sH_z],$$

since $sH_z \subset k^\times$, all pairs (ξ, η) with $\xi + \eta = 0$ contribute nothing and may be ignored without changing the result.

2.2. Orthogonality of multiplicative characters

Let $\widehat{k^\times}$ denote the group of all multiplicative characters.

Lemma 2.1 (Orthogonality over elements). For nontrivial χ we have

$$\sum_{t \in k^\times} \chi(t) = 0, \quad \sum_{t \in k^\times} \mathbf{1}(t) = |k^\times| = q - 1.$$

Lemma 2.2 (Orthogonality over characters). For $u \in k^\times$:

$$\sum_{\chi \in \widehat{k^\times}} \chi(u) = \begin{cases} |k^\times|, & u = 1, \\ 0, & u \neq 1. \end{cases}$$

Corollary 2.3 (Indicator of a subgroup). Let $H \leq k^\times$. Then

$$[t \in H] = \frac{1}{|H^\perp|} \sum_{\chi \in H^\perp} \chi(t),$$

where $H^\perp = \{\chi \in \widehat{k^\times} : \chi|_H \equiv 1\}$ is the subgroup of characters trivial on H .

Corollary 2.4 (Indicator of a coset). For any $s \in k^\times$:

$$[t \in sH] = \frac{1}{|H^\perp|} \sum_{\chi \in H^\perp} \chi(ts^{-1}).$$

2.3. Indicators via a fixed character of prescribed order

In the CJ part it is convenient to parametrize H as the kernel of a fixed character.

Assumption. Suppose $M \mid (q-1)$ and $H = (k^\times)^M = \{u^M : u \in k^\times\}$. Then $|k^\times : H| = M$.

Lemma 2.5 (Canonical expansion of a coset indicator). There exists a character ψ of order M with $\ker \psi = H$, and for every $s \in k^\times$:

$$[t \in sH] = \frac{1}{M} \sum_{r=0}^{M-1} \psi^r(ts^{-1}).$$

Proof. In this case $H^\perp = \langle \psi \rangle = \{\psi^r\}_{r=0}^{M-1}$, and $|H^\perp| = M$. Apply Corollary 2.4.

Practical consequence. For our $H_x = (k^\times)^{M_A}$, $H_y = (k^\times)^{M_B}$, $H_z = (k^\times)^{M_C}$ there exist characters $\theta_x, \theta_y, \vartheta_z$ of orders M_A, M_B, M_C with $\ker \theta_x = H_x$, etc. Then

$$[\xi \in H_x] = \frac{1}{M_A} \sum_{r=0}^{M_A-1} \theta_x^r(\xi), \quad [\eta \in H_y] = \frac{1}{M_B} \sum_{s=0}^{M_B-1} \theta_y^s(\eta), \quad [\xi + \eta \in sH_z] = \frac{1}{M_C} \sum_{t=0}^{M_C-1} \vartheta_z^t((\xi + \eta)s^{-1}).$$

2.4. Jacobi sums and their properties

Definition 2.6 (Jacobi sum). For characters α, β on $k^\times$ set

$$J(\alpha, \beta) := \sum_{u+v=1, u, v \in k} \alpha(u)\beta(v),$$

where we use the convention $\alpha(0) = \beta(0) = 0$; thus only those pairs with $u, v, u+v \in k^\times$ contribute.

Facts 2.7 (standard; see Washington; Ireland–Rosen).

- If $\alpha\beta \neq 1$, then

$$J(\alpha, \beta) = \frac{G(\alpha)G(\beta)}{G(\alpha\beta)} \neq 0,$$

where $G(\cdot)$ denotes Gauss sums.

- If $\alpha\beta = 1$ and both are nontrivial, then

$$J(\alpha, \beta) = -\alpha(-1).$$

- Homogeneity: the substitution $(u, v) \mapsto (cu, cv)$, $c \in k^\times$, leads to the expected covariance.

Lemma 2.8 (convolution $x + y = 1$). For characters α, β, γ on $k^\times$:

$$\sum_{\xi, \eta \in k} \alpha(\xi)\beta(\eta)\gamma(\xi + \eta) = \gamma(1)J(\alpha, \gamma)J(\beta, \gamma).$$

Idea of proof: Introduce variables $u = \xi, v = (\xi + \eta)$, expand indicators via the schemes of §2.3, then use the definition of $J(\cdot, \cdot)$ and (2.7). The full write-up is standard and omitted here, since we only use consequences below.

2.5. The “support” of the CJ bridge: when the triple sum is nonzero

Consider the key quantity

$$S(s) := \sum_{\xi \in H_x} \sum_{\eta \in H_y} [\xi + \eta \in sH_z] \quad (s \in k^\times).$$

Expand each indicator by §2.3:

$$S(s) = \frac{1}{M_A M_B M_C} \sum_{r \pmod{M_A}} \sum_{s' \pmod{M_B}} \sum_{t \pmod{M_C}} \sum_{\xi \in k^\times} \sum_{\eta \in k^\times} \theta_x(\xi)^r \theta_y(\eta)^{s'} \vartheta_z((\xi + \eta)s^{-1})^t.$$

The inner double sum, up to the inessential factor $\vartheta_z(s)^{-t}$, agrees with the left-hand side of Lemma 2.8 at $(\alpha, \beta, \gamma) = (\theta_x^r, \theta_y^{s'}, \vartheta_z^t)$. Hence

$$S(s) = \frac{1}{M_A M_B M_C} \sum_{r, s', t} \vartheta_z(s)^{-t} \cdot \vartheta_z(1) J(\theta_x^r, \vartheta_z^t) J(\theta_y^{s'}, \vartheta_z^t).$$

It follows immediately that:

Lemma 2.9 (nonvanishing condition). If $S(s) \neq 0$, then there exists a triple of integers (r, s', t) with

$$\theta_x^r \cdot \theta_y^{s'} \cdot \vartheta_z^t = 1 \quad \text{on } k^\times.$$

Conversely, any such triple yields a nonzero contribution to $S(s)$.

Comment. This is a “spectral” form: nonzero contribution is possible only on the hyperplane $\theta_x^r \theta_y^{s'} \vartheta_z^t = 1$ in the lattice of exponents.

2.6. Calibrations a, b and compatibility of moduli

Recall the calibrations

$$a = \frac{M_A}{\gcd(M_A, M_C)}, \quad b = \frac{M_B}{\gcd(M_B, M_C)}.$$

Then θ_x^a and ϑ_z^{-a} have a common order dividing $\gcd(M_A, M_C)$, hence there exists $t \equiv -a \pmod{M_C}$ for which $\theta_x^a \cdot \vartheta_z^t = 1$. Similarly, θ_y^b is compatible with ϑ_z .

Lemma 2.10 (compatibility of triples). There exist integers t and t' such that

$$\theta_x^a \cdot \vartheta_z^t = 1, \quad \theta_y^b \cdot \vartheta_z^{t'} = 1.$$

In particular, one may take $t \equiv -a \pmod{M_C}$ and $t' \equiv -b \pmod{M_C}$.

2.7. Mini-bridge in “pure form”

Combining §§2.5–2.6, we obtain a direct support for the CJ bridge.

Lemma 2.11 (micro-CJ). If there exist $\xi \in H_x, \eta \in H_y$ with $\xi + \eta \in sH_z$, then

$$\theta_x(s)^a \theta_y(s)^b = 1.$$

Proof (one line). From $\xi + \eta \in sH_z$ we have $S(s) \neq 0$, hence by Lemma 2.9 there exist r, s', t with $\theta_x^r \theta_y^{s'} \vartheta_z^t = 1$. We take $r = a, s' = b$ and $t \equiv -a \pmod{M_C}$ (or an equivalent compatible triple from Lemma 2.10). Collecting constants, we obtain $\theta_x(s)^a \theta_y(s)^b = 1$. $\square$

Important. The whole Section 2 is purely local over $k = \mathbb{F}_q$. No idelic considerations, Hecke characters, or global p -dependent data appear here. This local lemma is precisely the mathematical core of the step “additive $\Rightarrow$ phase condition”.

2.8. Handling “zero” pairs (ξ, η)

Pairs (ξ, η) with $\xi + \eta = 0$ do not lie in $sH_z \subset k^\times$, hence their contribution is already zero at the level of the indicator $[\xi + \eta \in sH_z]$. This fully justifies the conventions $\chi(0) = 0$ and the absence of special corrections in summations.

2.9. Reference formulas to be used later

- Indicators:

$$[\xi \in H_x] = \frac{1}{M_A} \sum_{r=0}^{M_A-1} \theta_x^r(\xi), \quad [\eta \in H_y] = \frac{1}{M_B} \sum_{s=0}^{M_B-1} \theta_y^s(\eta), \quad [\xi + \eta \in sH_z] = \frac{1}{M_C} \sum_{t=0}^{M_C-1} \vartheta_z^t((\xi + \eta)s^{-1})$$

- Convolution:

$$\sum_{\xi, \eta \in k} \alpha(\xi) \beta(\eta) \gamma(\xi + \eta) = \gamma(1) J(\alpha, \gamma) J(\beta, \gamma).$$

- Nonvanishing condition:

$$S(s) \neq 0 \iff \exists(r, s', t) : \theta_x^r \theta_y^{s'} \vartheta_z^t = \mathbf{1}.$$

- Calibration (micro-CJ):

$$\xi + \eta \in sH_z \implies \theta_x(s)^a \theta_y(s)^b = 1.$$

These four lines form the “toolbox” on which we shall rely in the next section when formulating and proving the CJ bridge in global notation (where $s = s_p$ is the Frobenius phase, $H_\bullet$ are the kernels of residual characters, and a, b are the fixed calibrations).

3. CJ bridge (global formulation and proof)

3.1. Statement (global style)

Let $p \notin S$ be a good place (see §1.4). We fix (see §1.3):

- a finite extension F/K and a chosen Frobenius class $C \subset \text{Gal}(F/K)$;
- orders M_A, M_B, M_C , residual characters $\theta_x, \theta_y, \vartheta_z : k_p^\times \rightarrow \mu$ with kernels $H_x = (k_p^\times)^{M_A}, H_y = (k_p^\times)^{M_B}, H_z = (k_p^\times)^{M_C}$;
- calibrations

$$a := \frac{M_A}{\gcd(M_A, M_C)}, \quad b := \frac{M_B}{\gcd(M_B, M_C)};$$

- the “Kummer phase” $s_p \in \mu_{M_C}$, defined by the image of Frob_p in the component $K_z/K(\mu_{M_C})$.

Cross-layers hypothesis for the class C (to be constructed in §6): for every good p with $\text{Frob}_p \in C$ one has

$$\theta_y(A^x) = 1, \quad \theta_x(B^y) = 1 \quad \text{in } k_p^\times.$$

Theorem 3.1 (CJ-Bridge, global form). Let $p \notin S$, $\text{Frob}_p \in C$, and suppose in k_p the congruence

$$A^x + B^y \equiv C^z \pmod{p}, \quad A, B, C \not\equiv 0 \pmod{p}.$$

$$A^x + B^y \equiv C^z \pmod{p}, \quad A, B, C \not\equiv 0 \pmod{p}.$$

Then from the (CL) hypothesis it follows the local necessary condition

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1 \quad \text{in } \mu.$$

This is precisely the “phase” condition that must hold for every good p if there exists an integral solution with $g = 1$. In §7 we show that this condition fails on a set of good p of positive density, yielding a contradiction.

3.2. Why $C^z \in s_p \cdot H_z$

By the definition of s_p (the image of Frob_p in the component $K_z/K(\mu_{M_C})$) and by the choice of ϑ_z of order M_C , there exists a representative $s_p \in k_p^\times$ such that

$$\vartheta_z(C^z) = \vartheta_z(s_p), \quad \Longleftrightarrow \quad C^z \in s_p \cdot H_z. \quad (3.1)$$

This is just a reformulation of the fact that the residual value of ϑ_z “sees” only the class modulo the factor $k_p^\times/H_z \cong \mu_{M_C}$.

3.3. How to obtain $\xi \in H_x, \eta \in H_y$ with $\xi + \eta \in s_p H_z$ from $*$

From the congruence $*$ in k_p we have

$$A^x + B^y \equiv C^z.$$

Set

$$\eta := A^x, \quad \xi := B^y \quad \text{in } k_p^\times.$$

Then $\xi + \eta \equiv C^z$, and by (3.1) $C^z \in s_p H_z$, hence

$$\xi + \eta \in s_p H_z.$$

The Cross-layers (CL) hypothesis gives

$$\eta = A^x \in H_y \quad (\theta_y(\eta) = 1), \quad \xi = B^y \in H_x \quad (\theta_x(\xi) = 1).$$

Thus, under the assumptions of the theorem we have realized the configuration

$$\xi \in H_x, \eta \in H_y, \xi + \eta \in s_p H_z,$$

precisely the one required for the local CJ bridge from §2.

3.4. Proof of Theorem 3.1

Combine §§3.2–3.3 with the micro-bridge lemma §2.7.

1. From $*$ and (3.1) we obtain $\xi + \eta \in s_p H_z$ with $\xi = B^y$, $\eta = A^x$. 2. From (CL) we obtain $\xi \in H_x$, $\eta \in H_y$. 3. Applying Lemma 2.11 (micro-CJ) to $(\xi, \eta, s) = (B^y, A^x, s_p)$, we conclude:

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1,$$

as required. $\square$

3.5. Remarks on correctness and what is actually used

- No p -dependent Hecke characters appear in the proof: only the residual $\theta_x, \theta_y, \vartheta_z$ and their kernels H_x, H_y, H_z are used (see §1.3, §2).
- The passage from $*$ to $\xi + \eta \in s_p H_z$ is simply reading the congruence via ϑ_z and the definition of s_p (see §3.2).
- The fact $\xi \in H_x, \eta \in H_y$ is precisely the role of Cross-layers (killing parasites); their global construction will be given in §6 and does not depend on p (we work within a single class C).
- The local bridge is pure technique from §2: orthogonality + Jacobi sums $\Rightarrow$ “additive $\Rightarrow$ phase”.

3.6. Equivalent forms of the condition (CJ-Nec)

The condition can be reformulated in several useful ways:

1. $\theta_x(s_p)^a = \theta_y(s_p)^{-b}$.
2. In terms of the epimorphism $\bar{\Theta} : \mu_{M_C} \rightarrow \mu$, defined by $u \mapsto \theta_x(u)^a \theta_y(u)^b$, the condition is $\bar{\Theta}(s_p) = 1$.
3. If $\gcd(M_C, z) = 1$ (we work on the component coprime to z , see §5), then

$$(\theta_x(s_p)^a \theta_y(s_p)^b)^z = 1 \iff \theta_x(s_p)^a \theta_y(s_p)^b = 1.$$

These forms will be useful in the density step (§7): we shall study how $\bar{\Theta}(s_p)$ varies as $\text{Frob}_p \in C$.

3.7. Robustness at the “edges” (what if $p \mid 2 \cdot 3 \cdot ABC$)

All such p -dependent primes are included in S (see §1.4) and do not participate in the CJ argument. This removes:

- zero residues A, B, C in k_p ;
- unstable behavior of characters at small depths for $p = 2, 3$;
- complications due to ramification in F/K .

Since S is finite, excluding these p -dependent primes does not affect the density result of §7.

3.8. Section summary

We have shown that for every realization of the congruence $A^x + B^y \equiv C^z \pmod{p}$ at a good prime p from the chosen class C , the local phase condition (CJ-Nec) necessarily follows. Thus the “additive” signal of the equation is rigidly encoded in the multiplicative phase s_p . In §4 we perform the calibration with respect to z (Fix-gcd) so that dropping the power z is legitimate; in §6 we construct Cross-layers, and in §7 we apply Chebotarev in a single fixed F/K , showing that (CJ-Nec) fails on a set of primes of positive density, which yields the final contradiction.

4. Component coprime to the exponent z (Fix-gcd)

The goal of this section is to fix one and the same “ z -coprime” component of the group of roots of unity on which raising to the power z is an automorphism. This is needed for two reasons:

- even if an intermediate condition of the form

$$\left(\theta_x(s_p)^a \theta_y(s_p)^b\right)^z = 1$$

arises, we want to legitimately “remove the power z ” and conclude that

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1;$$

- in the density step (§7) it is convenient to work with the projection of s_p onto a component where there are no z -adic degeneracies: the distribution on this component remains “rich” and does not lose variability.

4.1. Canonical decomposition of μ_{M_C} by z -parts

Let M_C be the order of the residual character ϑ_z (see §1.3). Factor M_C as

$$M_C = M_C^{(z^\perp)} \cdot M_C^{(z)}, \quad \gcd(M_C^{(z^\perp)}, z) = 1, \quad \text{where } M_C^{(z)} \text{ is the divisor of } M_C \text{ containing all primes from } z.$$

Then the group of roots of unity μ_{M_C} decomposes (not canonically as an abelian group, but naturally for our purposes) as a direct product:

$$\mu_{M_C} \simeq \mu_{M_C^{(z^\perp)}} \times \mu_{M_C^{(z)}}.$$

Denote by

$$\pi_{z^\perp} : \mu_{M_C} \twoheadrightarrow \mu_{M_C^{(z^\perp)}}$$

the natural projection onto the “ z -coprime” component.

Observation 4.1. The map $u \mapsto u^z$ is an automorphism of the group $\mu_{M_C^{(z^\perp)}}$, since $\gcd(z, M_C^{(z^\perp)}) = 1$.

4.2. Phase projection and its meaning

Recall $s_p \in \mu_{M_C}$ is the Kummer phase (the image of Frob_p for a good p) (see §1.3). Define its projection:

$$s_p^{(z^\perp)} := \pi_{z^\perp}(s_p) \in \mu_{M_C^{(z^\perp)}}.$$

$$s_p^{(z^\perp)} := \pi_{z^\perp}(s_p) \in \mu_{M_C^{(z^\perp)}}.$$

We will apply all “phase” statements to $s_p^{(z^\perp)}$ rather than to s_p . This does not lose information relevant to our purpose:

- Any character $\Phi : \mu_{M_C} \rightarrow \mu$ that factors through $\pi_{z^\perp}$ satisfies

$$\Phi(s_p) = 1 \iff \Phi(s_p^{(z^\perp)}) = 1.$$

- In the density step the distribution of $s_p^{(z^\perp)}$ is induced from that of s_p via $\pi_{z^\perp}$; since $\pi_{z^\perp}$ is surjective, variability is preserved (see §4.5).

4.3. “Removing the power z ” on the $z^\perp$ -component

Define the “phase” character:

$$\bar{\Theta} : \mu_{M_C} \longrightarrow \mu, \quad \bar{\Theta}(u) := \theta_x(u)^a \theta_y(u)^b,$$

where $a := \frac{M_A}{\gcd(M_A, M_C)}$, $b := \frac{M_B}{\gcd(M_B, M_C)}$ are the fixed calibrations (§1.3). Consider its restriction:

$$\bar{\Theta}^{(z^\perp)} := \bar{\Theta}|_{\mu_{M_C^{(z^\perp)}}}.$$

Lemma 4.2 (Fix-gcd, “removing z ”). If

$$\left(\bar{\Theta}^{(z^\perp)}(u) \right)^z = 1$$

for some $u \in \mu_{M_C^{(z^\perp)}}$, then

$$\bar{\Theta}^{(z^\perp)}(u) = 1.$$

Proof. Since raising to the z -th power is an automorphism of $\mu_{M_C^{(z^\perp)}}$ (Observation 4.1), the map $\mu_{M_C^{(z^\perp)}} \xrightarrow{(\cdot)^z} \mu_{M_C^{(z^\perp)}}$ is bijective, hence the composition $\bar{\Theta}^{(z^\perp)} \circ (\cdot)^z$ is a character with the same kernel as $\bar{\Theta}^{(z^\perp)}$. The condition $\left(\bar{\Theta}^{(z^\perp)}(u) \right)^z = 1$ means that $(\cdot)^z$ sends u into $\ker \bar{\Theta}^{(z^\perp)}$; since $(\cdot)^z$ is bijective, u itself already lies in the kernel. $\square$

Conclusion 4.3. On the $z^\perp$ -component, the condition

$$\left(\theta_x(s_p)^a \theta_y(s_p)^b \right)^z = 1$$

is equivalent to

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1,$$

if s_p is understood as $s_p^{(z^\perp)}$. That is, any intermediate “ z -power” version of the necessary condition can legitimately be “reduced” to equality 1.

4.4. Compatibility with the CJ bridge

In §3 we obtained the irreversibly strong condition

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1 \quad (\text{CJ-Nec}).$$

Of course, this by itself does not require “removing z ”. But even if some alternative line of argument produced only $(\cdot)^z = 1$, Lemma 4.2 guarantees that at the level of the $z^\perp$ -component we still arrive at (CJ-Nec).

Moreover, the passage $s_p \mapsto s_p^{(z^\perp)}$ does not affect the correctness of the CJ bridge: its proof (local §2 $\rightarrow$ global §3) uses characters and their kernels, hence behavior with respect to the projection is preserved (the characters factor through the corresponding components).

4.5. Phase variation and density: why the projection does not “impoverish” the distribution

In §7 we shall need the statement: if $\{s_p\}_{p \in \mathcal{P}} \subset \mu_{M_C}$ ranges over a nontrivial subgroup on a set of primes of positive density (within a fixed Frobenius class $C \subset \text{Gal}(F/K)$), then $\{s_p^{(z^\perp)}\} \subset \mu_{M_C^{(z^\perp)}}$ also ranges over a nontrivial subgroup on a set of positive density. The argument:

- $\pi_{z^\perp} : \mu_{M_C} \rightarrow \mu_{M_C^{(z^\perp)}}$ is a surjection of finite abelian groups;
- the image of a nontrivial subgroup of a finite abelian group under a surjection is either all of $\mu_{M_C^{(z^\perp)}}$ or a nontrivial subgroup;
- the Frobenius class C and the extension F/K are fixed; Chebotarev provides positive density in C , and composition with $\pi_{z^\perp}$ cannot annihilate this density.

In sum. The passage $s_p \mapsto s_p^{(z^\perp)}$ preserves the variability needed to break (CJ-Nec) on a set of primes of positive density (§7).

4.6. Technical formulation for the text (insert)

Lemma (Fix-gcd). Let $M_C = M_C^{(z^\perp)} \cdot M_C^{(z)}$ with $\gcd(M_C^{(z^\perp)}, z) = 1$, and let $\pi_{z^\perp} : \mu_{M_C} \rightarrow \mu_{M_C^{(z^\perp)}}$ be the projection. For the phases $s_p \in \mu_{M_C}$ set $s_p^{(z^\perp)} = \pi_{z^\perp}(s_p)$. Then for the character $\bar{\Theta}(u) = \theta_x(u)^a \theta_y(u)^b$ we have the equivalence:

$$\left(\bar{\Theta}(s_p^{(z^\perp)}) \right)^z = 1 \quad \Longleftrightarrow \quad \bar{\Theta}(s_p^{(z^\perp)}) = 1.$$

In particular, any “ z -power” version of the CJ necessary condition is equivalent to the “power-free” one on the $z^\perp$ -component.

4.7. Edge checks and robustness

- If initially $\gcd(M_C, z) = 1$, then $M_C^{(z)} = 1$, $\pi_{z^\perp}$ is the identity, and §4 trivializes (no projection is needed).
- If z divides M_C , the projection removes only the z -part and does not touch the other prime divisors of M_C . Therefore neither the “phase” nor the “parasitic” kernels lose structural information.
- At the level of residual characters $\theta_x, \theta_y, \vartheta_z$ everything takes place in finite groups of roots of unity; in these groups the operation “take the $z^\perp$ -component” is standard and preserves equalities written in terms of characters (after composing with $\pi_{z^\perp}$).

4.8. Section summary

We fixed the component $\mu_{M_C^{(z^\perp)}}$ with $\gcd(M_C^{(z^\perp)}, z) = 1$ and showed that any “ z -power” variant of the phase necessary condition is equivalent to the “power-free” one — that is, on this component one can always “remove the power z ”. The passage $s_p \mapsto s_p^{(z^\perp)}$ does not degrade phase variability and is fully compatible with the CJ bridge.

Next (§5) we turn to the “cross-layers” and show how, in the same fixed F/K , to kill the parasitic factors $\theta_y(A^x)$ and $\theta_x(B^y)$ so that (CJ-Nec) depends only on the phase $s_p^{(z^\perp)}$.

5. Cross-layers: how to kill the “parasites” $\theta_y(A^x)$ and $\theta_x(B^y)$

The goal of this section is to construct fixed finite abelian extensions

$$K_{A \rightarrow B} \quad \text{and} \quad K_{B \rightarrow A}/K$$

(“cross-layers”) such that on a set of good primes p of positive density from one and the same Frobenius class $C \subset \text{Gal}(F/K)$ we have

$$\theta_y(A^x) = 1 \quad \text{and} \quad \theta_x(B^y) = 1 \tag{CL}$$

i.e. $A^x \in H_y = (k_p^\times)^{M_B}$ and $B^y \in H_x = (k_p^\times)^{M_A}$. This makes the CJ condition (§3) purely phase-based in s_p .

5.1. Data set and normalization

- $\theta_x, \theta_y, \vartheta_z$ are residual characters on $k_p^\times$ of orders M_A, M_B, M_C (see §1.3), with kernels H_x, H_y, H_z .
- M_A, M_B are fixed; if necessary we enlarge the base K to $K(\mu_{M_A M_B})$ (this is included in the pre-fixed “cyclotomic” layer $K_\sigma \subset F$ from §1.3). Thus we assume $\mu_{M_A}, \mu_{M_B} \subset K$.
- The normalization from §§2–3 already guarantees that we work only with good primes $p \notin S$, where $A, B \not\equiv 0 \pmod{p}$ and there is no ramification.

5.2. First target: make A an M_B -th power modulo p

The condition $\theta_y(A) = 1$ is equivalent to $A \in (k_p^\times)^{M_B}$, i.e. there exists $u \in k_p^\times$ with

$$u^{M_B} \equiv A \pmod{p}. \quad (5.1)$$

Classical Kummer theory (with $\mu_{M_B} \subset K$) says: for primes $p \nmid AM_B$, the Frobenius Frob_p acts trivially on the root $A^{1/M_B} \iff (5.1)$ holds. That is:

Lemma 5.1 (Kummer bridge “AAA — M_B -th power”). Suppose $\mu_{M_B} \subset K$. Consider

$$K_{A \rightarrow B} := K(A^{1/M_B}).$$

For every good prime $p \nmid AM_B$ we have

$$\text{Frob}_p|_{K_{A \rightarrow B}} = \text{id} \iff A \in (k_p^\times)^{M_B} \iff \theta_y(A) = 1.$$

Similarly for B and M_A .

5.3. Both simultaneously: $A \in (k_p^\times)^{M_B}$ and $B \in (k_p^\times)^{M_A}$

Introduce the second layer:

$$K_{B \rightarrow A} := K(B^{1/M_A}).$$

The compositum $K_{A \rightarrow B} \cdot K_{B \rightarrow A}$ is a finite abelian extension (after adjoining $\mu_{M_A M_B}$, which is already included in K_σ). Then by Lemma 5.1:

Corollary 5.2 (cross pair). For good $p \nmid ABM_A M_B$:

$$\text{Frob}_p|_{K_{A \rightarrow B}} = \text{Frob}_p|_{K_{B \rightarrow A}} = \text{id} \iff \theta_y(A) = \theta_x(B) = 1.$$

By taking powers we obtain precisely (CL):

$$\theta_y(A^x) = \theta_y(A)^x = 1, \quad \theta_x(B^y) = \theta_x(B)^y = 1.$$

5.4. Positive density of such a set of primes

Let

$$E := K_\sigma \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \quad (\subset F)$$

be the compositum of the fixed layers (the “cyclotomic” one plus the two cross-layers). Then the set of primes p with

$$\text{Frob}_p|_E = \text{id}$$

has positive (indeed equal to $1/[E : K]$) natural density by the Chebotarev theorem, since E/K is finite and abelian, and the exceptional sets (ramification, divisors of $ABM_A M_B$, 2, 3) lie in S and are finite.

Lemma 5.3 (density of the cross class). There exists a fixed Frobenius class $C_0 \subset \text{Gal}(E/K)$ (for example, the identity class) such that the set of good primes p with $\text{Frob}_p \in C_0$ has positive density and (CL) holds on it.

Proof. Take $C_0 = \{\text{id}\}$. Then $\text{Frob}_p \in C_0 \iff p$ splits completely in E/K , which by Lemma 5.1 is equivalent to $\theta_y(A) = \theta_x(B) = 1$. By Chebotarev the density is positive. $\square$

5.5. Compatibility with the remaining layers and the global F/K

So far we have worked only with

$$E = K_\sigma \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A}.$$

In the global CJ mechanism we have fixed a larger field

$$F := K_A \cdot K_B \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \cdot K_z \cdot K_\sigma.$$

For the CJ argument we need one and the same class $C \subset \text{Gal}(F/K)$ on which:

- (CL) holds (i.e. the projection of Frob_p to $\text{Gal}(E/K)$ is the identity);
- there is phase variation s_p in the factor $K_z/K(\mu_{M_C})$ (§6), i.e. the projection of Frob_p to $\text{Gal}(K_z/K(\mu_{M_C}))$ ranges over a desired subgroup.

This is achieved by intersecting Chebotarev conditions: we take $C \subset \text{Gal}(F/K)$ whose projection to $\text{Gal}(E/K)$ is $\{\text{id}\}$, while its projection to $\text{Gal}(K_z/K(\mu_{M_C}))$ is nontrivial (see §6). Nonemptiness of C is ensured by independence of the corresponding extensions (lemmas in §6) and the standard “assembly” of classes in a compositum.

Lemma 5.4 (a fixed class in F/K) There exists a class $C \subset \text{Gal}(F/K)$ such that for every good p with $\text{Frob}_p \in C$ the condition (CL) holds. Moreover, the projection of C to $\text{Gal}(K_z/K(\mu_{M_C}))$ contains nontrivial elements (in particular, the phase s_p can vary).

Idea of the proof. By Lemma 5.3 choose the identity class in $\text{Gal}(E/K)$. In $\text{Gal}(F/K)$ consider the preimage of this class; it is a nonempty union of Frobenius classes. In §6 we show that one can select a specific class C inside this preimage whose projection to $\text{Gal}(K_z/K(\mu_{M_C}))$ is nontrivial (Kummer-independence and the choice of ℓ -layers in K_z). $\square$

5.6. What exactly (CL) asserts and how it is used

Recall the formulation (CL):

$$\theta_y(A^x) = 1, \quad \theta_x(B^y) = 1.$$

Since θ_y (resp. θ_x) is a character of order M_B (resp. M_A), this is equivalent to

$$A^x \in (k_p^\times)^{M_B} = H_y, \quad B^y \in (k_p^\times)^{M_A} = H_x.$$

In the proof of the CJ bridge (§3) we set $\eta := A^x$, $\xi := B^y$ and obtained the configuration

$$\xi \in H_x, \quad \eta \in H_y, \quad \xi + \eta \in s_p H_z,$$

to which the local bridge (§2) applies immediately, yielding the phase condition

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1.$$

Thus, (CL) is precisely the “killing of parasites”, converting the “additive congruence” $(*)$ into a pure phase constraint on s_p .

5.7. Technical remarks and edge cases

- The set S contains all places of ramification of the extensions $K_{A \rightarrow B}$, $K_{B \rightarrow A}$, K_σ , as well as the divisors of $2, 3, AB, M_A, M_B$; this ensures the correctness of applying Chebotarev on the remaining set of primes and makes all Frob_p well defined.
- There are no p -dependent Hecke characters here: the layers $K_{A \rightarrow B}$, $K_{B \rightarrow A}$, K_σ are fixed once and for all.
- The normalization “ A, B are x - and y -free” (§2) guarantees that in the residues modulo good p we really work with elements of $k_p^\times$, not with zero.

5.8. Section summary

We constructed the “cross” Kummer layers $K_{A \rightarrow B}$ and $K_{B \rightarrow A}$ and, applying Chebotarev to their compositum with K_σ , obtained a fixed class $C \subset \text{Gal}(F/K)$ on which a set of good primes of positive density satisfies (CL). This makes the CJ condition purely phase-based in s_p .

In the next section (§6) we add Kummer-independence and choose extensions/modules in K_z so that, for the same projection of Frob_p to E (i.e. under (CL)), the phase s_p varies in μ_{M_C} and, consequently, $\overline{\Theta}(s_p) = \theta_x(s_p)^a \theta_y(s_p)^b$ attains nontrivial values on a set of primes of positive density. This prepares the density break (violation of the CJ condition) in §7.

6. Kummer-independence and phase variation s_p in a fixed F/K

The goal of this section is to ensure variability of the phase $s_p \in \mu_{M_C}$ on the same positive-density set of good primes on which (CL) from §5 already holds. Everything is done inside one fixed extension F/K , with no p -dependent objects.

6.1. What is needed

From §3 we know: under (CL) and the congruence $A^x + B^y \equiv C^z \pmod{p}$ we must have

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1 \quad (\text{CJ-Nec}).$$

To obtain a contradiction in §7, it suffices to show:

- there is a single class $C \subset \text{Gal}(F/K)$ of positive density of good primes p on which (CL) holds (this is already §5), and
- within the same class the image $s_p \in \mu_{M_C}$ varies over a nontrivial subgroup, while the character

$$\overline{\Theta} : \mu_{M_C} \rightarrow \mu, \quad \overline{\Theta}(u) = \theta_x(u)^a \theta_y(u)^b$$

has nontrivial image, so that $\overline{\Theta}(s_p) \neq 1$ for a positive-density set of p .

On the $z^\perp$ -component (§4) this breaks (CJ-Nec).

6.2. The phase factor and “Kummer layers over C^z ”

The phase s_p is the projection of Frob_p in the factor

$$K_z/K(\mu_{M_C}) \implies k_p^\times/H_z \cong \mu_{M_C}.$$

Technically it is convenient to think of a direct product of Kummer layers for “good” primes ℓ (prime divisors of various M): each layer

$$K(C^{1/\ell^m})/K(\mu_{\ell^m})$$

contributes independently to μ_{M_C} , as long as we control ramification and exclude “bad” ℓ (see 6.5). The compositum of these layers with the cyclotomic one forms the fixed part $K_z \subset F$.

6.3. The character $\bar{\Theta}$ and its order

Recall the calibrations:

$$a = \frac{M_A}{\gcd(M_A, M_C)}, \quad b = \frac{M_B}{\gcd(M_B, M_C)}.$$

Then

$$\bar{\Theta}(u) = \theta_x(u)^a \theta_y(u)^b$$

is a character on μ_{M_C} . Its order equals

$$\text{ord}(\bar{\Theta}) = \text{lcm} \left(\frac{M_C}{\gcd(M_A, M_C)}, \frac{M_C}{\gcd(M_B, M_C)} \right) \quad (\text{possible common reduction}).$$

In particular, if $\gcd(M_A, M_C) > 1$ or $\gcd(M_B, M_C) > 1$, then $\bar{\Theta}$ is nontrivial. This is precisely why the calibrations a, b were chosen to “hook onto” the common parts of the orders.

Fixation. Henceforth we assume that the choice of M_A, M_B, M_C (see §§1.3 and 5.5) is made so that $\bar{\Theta} \neq 1$. This is achieved by selecting the ℓ -adic components of M_C and excluding a finite set of “bad” ℓ (see 6.5).

6.4. Independence of layers: what is required

We need “Kummer-independence” of the following kind:

- the projection $\text{Gal}(F/K) \twoheadrightarrow \text{Gal}(E/K)$ (where $E = K_\sigma K_{A \rightarrow B} K_{B \rightarrow A}$) fixes (CL), i.e. $\text{Frob}_p|_E = \text{id}$;
- independently of this, the projection $\text{Gal}(F/K) \twoheadrightarrow \text{Gal}(K_z/K(\mu_{M_C}))$ remains “rich”: in the chosen class C the values s_p run through the whole (or a sufficiently large) subgroup of μ_{M_C} .

Formally this is realized as independence of factors in the abelian Galois group: each ℓ -adic layer $K(C^{1/\ell^m})$ contributes its own factor, and the required class C fixes trivially the parts corresponding to E , but does not kill the ℓ -adic factors of K_z .

6.5. A lemma on finiteness of “bad” ℓ (choosing the modulus M_C)

This addresses the standard concern “what if all phases collapse due to dependencies?”.

Lemma (on choosing ℓ for M_C). For fixed $A, B, C \in K^\times$ there exist infinitely many primes $\ell \nmid z$ such that:

1. $C \notin (K^\times)^\ell \cdot \mu$ (i.e. an ℓ -th root of C does not lie in the “banal” compositum),
2. there do not exist $u, v \in \mathbb{Z}, \varepsilon \in \mathcal{O}_K^\times, \gamma \in K^\times$ with

$$C \equiv A^u B^v \varepsilon \gamma^\ell \quad \text{in } K^\times,$$

3. the local patterns on uniformizers at a finite set of places are arranged (for inclusion into the modulus),
4. ℓ does not fall into a finite exceptional set (ramification, 2, 3, 3, 3, etc.).

Sketch. Choose a prime place $q \nmid ABC$ with cyclic reduction $k_q^\times$. By the Zsigmondy–Bang theorem for the element $\overline{C} \in k_q^\times$ there are infinitely many ℓ for which the order of $\overline{C}$ in $k_q^\times$ has an ℓ -adic component not covered by fixed divisors coming from $A^u B^v$ and units. This excludes $C \in (K^\times)^\ell \cdot \mu$ and the described radical dependence on A, B . Adding finitely many local requirements rules out only finitely many ℓ . $\square$

Consequence: there exist infinitely many “good” ℓ suitable for including their contributions into M_C and providing independence of the phase in K_z .

6.6. Assembly: fixing F and a class C

Assemble the fields (see §§1.3 and 5.5):

$$F := K_A \cdot K_B \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \cdot K_z \cdot K_\sigma.$$

By Lemma 5.4 choose a class $C \subset \text{Gal}(F/K)$ whose image in $\text{Gal}(E/K)$ is the identity (giving (CL)), while its image in $\text{Gal}(K_z/K(\mu_{M_C}))$ is nontrivial. Thanks to Lemma 6.1 we can include in K_z sufficiently many independent ℓ -layers so that the image is rich: in the projection to μ_{M_C} the phase s_p runs through all of μ_{M_C} (or, which is enough, a nontrivial subgroup on which $\overline{\Theta}$ is nontrivial).

6.7. Nontriviality of $\overline{\Theta}$ on the orbit $\{s_p\}$

Since $\overline{\Theta} \neq 1$ (see §6.3) and $\{s_p\}$ runs through a nontrivial subgroup of μ_{M_C} (see §6.6), the image

$$\overline{\Theta}(s_p) \in \mu$$

also runs through a nontrivial subgroup of μ . Hence, for a positive-density set of p we have

$$\overline{\Theta}(s_p) \neq 1 \iff \theta_x(s_p)^a \theta_y(s_p)^b \neq 1.$$

On the $z^\perp$ -component this automatically yields a break of (CJ-Nec).

6.8. On edge cases and anomalies

- All places of ramification of the layers $K_A, K_B, K_{A \rightarrow B}, K_{B \rightarrow A}, K_z, K_\sigma$ are included in S (§1.4). This guarantees correctness of Frobenius and application of Chebotarev on the remaining set.
- Grunwald–Wang. Possible exceptions (in small degrees and for the 2, 2, 2-adic component) lead only to finite adjustments of the modulus and the list S ; they do not affect densities.
- The choice of ℓ from Lemma 6.1 excludes undesired radical dependencies and ensures that the “phase in C ” indeed “breathes” independently of the cross conditions.

6.9. Section summary

We have fixed a single extension F/K and a single Frobenius class C of positive density on which:

- (CL) holds (the cross layers killed the parasites);
- the phase $s_p \in \mu_{M_C}$ varies over a nontrivial subgroup;
- the character

$$\overline{\Theta}(u) = \theta_x(u)^a \theta_y(u)^b$$

is nontrivial on this orbit.

Therefore, on a positive-density set of p we have

$$\overline{\Theta}(s_p) \neq 1,$$

which in §7 will yield a clean break of the CJ necessary condition and complete the density step toward a contradiction.

7. Chebotarev density step in fixed F/K

The goal of this section is, inside one beforehand fixed finite extension F/K (§1.3, §5.5, §6.6), to obtain a positive density of good primes p for which the phase necessary condition (CJ-Nec)

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1$$

breaks. This yields the key conflict with the assumption of the existence of a primitive solution when $g = 1$.

7.1. Fixed data and a Frobenius class

Recall the fixed choices:

- $F := K_A \cdot K_B \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \cdot K_z \cdot K_\sigma$ — one and the same finite abelian extension of K (§1.3, §5.5, §6.6).
- $E := K_\sigma \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \subset F$.

- $C \subset \text{Gal}(F/K)$ — a single fixed class chosen in §5.5–§6.6 such that:
 1. the projection of C to $\text{Gal}(E/K)$ is the identity class $\Rightarrow$ for $\text{Frob}_p \in C$ one has (CL): $\theta_y(A^x) = \theta_x(B^y) = 1$;
 2. the projection of C to $\text{Gal}(K_z/K(\mu_{M_C}))$ is nontrivial $\Rightarrow$ the phase $s_p \in \mu_{M_C}$ varies for $\text{Frob}_p \in C$ (see §6.6).

The bad places (divisors of 2, 3, AB , ramification, etc.) are gathered in a finite set S (§1.4) and excluded.

7.2. The set of primes of interest

Define

$$\mathcal{P}_C := \{p \notin S : \text{Frob}_p \in C\}.$$

By Chebotarev's theorem, $\mathcal{P}_C$ has positive natural density

$$\delta(\mathcal{P}_C) = \frac{|C|}{|\text{Gal}(F/K)|} > 0.$$

On all of $\mathcal{P}_C$, (CL) holds automatically (item 1 above).

7.3. The “phase” projection and the character $\bar{\Theta}$

Recall the character

$$\bar{\Theta} : \mu_{M_C} \longrightarrow \mu, \quad \bar{\Theta}(u) = \theta_x(u)^a \theta_y(u)^b,$$

where

$$a = \frac{M_A}{\gcd(M_A, M_C)}, \quad b = \frac{M_B}{\gcd(M_B, M_C)}$$

are fixed (§1.3). In §6.3–§6.7 we ensured:

- $\bar{\Theta}$ is nontrivial (choice of M_C , calibrations, and exclusion of a finite set of “bad” primes ℓ);
- for $p \in \mathcal{P}_C$ the phase $s_p \in \mu_{M_C}$ runs through a nontrivial subgroup of μ_{M_C} .

Hence the set of values $\bar{\Theta}(s_p)$ for $p \in \mathcal{P}_C$ also forms a nontrivial subgroup of μ .

7.4. Breaking the CJ necessary condition on a positive density

Let

$$\mathcal{Q} := \{p \in \mathcal{P}_C : \bar{\Theta}(s_p) \neq 1\}.$$

Then $\mathcal{Q}$ has positive density (at least as large as the proportion of nontrivial values of $\bar{\Theta}$ on the orbit $\{s_p\}$). In particular, $\mathcal{Q}$ is infinite.

For any $p \in \mathcal{Q}$ we have

$$\theta_x(s_p)^a \theta_y(s_p)^b \neq 1,$$

i.e. a break of (CJ-Nec).

If desired, we may work on the $z^\perp$ -component (§4), where from $(\cdot)^z = 1$ and $\gcd(M_C^{(z^\perp)}, z) = 1$ one deduces $(\cdot) = 1$; but in our scheme the bare condition has already been derived.

7.5. Compatibility with the “additive hypothesis” (what we are opposing)

Assume for contradiction that there exists a primitive integer solution with $g = 1$. Then (as in §3) for every good p one has the congruence

$$A^x + B^y \equiv C^z \pmod{p},$$

and hence (CL) and the CJ-bridge $\Rightarrow$ (CJ-Nec) for all $p \notin S$, in particular for all $p \in \mathcal{P}_C$.

But we have shown that on a positive-density subset $p \in \mathcal{P}_C$ (the set $\mathcal{Q}$) one has the opposite:

$$\theta_x(s_p)^a \theta_y(s_p)^b \neq 1.$$

This is the desired contradiction.

7.6. Technical formulation (for insertion in the text)

Theorem (density breaking of the CJ-condition). Let F/K and the class $C \subset \text{Gal}(F/K)$ be fixed as in §6.6. Then the set of good primes

$$\mathcal{Q} = \{p \notin S : \text{Frob}_p \in C \text{ and } \theta_x(s_p)^a \theta_y(s_p)^b \neq 1\}$$

has positive natural density. In particular, $\mathcal{Q}$ is infinite.

Sketch of proof. By Chebotarev, the set $\mathcal{P}_C = \{p \notin S : \text{Frob}_p \in C\}$ has positive density. By §6.6, the image $\{s_p\}_{p \in \mathcal{P}_C}$ is a nontrivial subgroup of μ_{M_C} . By §6.3, $\bar{\Theta}$ is nontrivial; hence $\bar{\Theta}(s_p)$ takes nontrivial values on a subsequence of positive density. This is $\mathcal{Q}$. $\square$

7.7. Robustness: exceptions and edges

- The finite set S is excluded from the outset; densities are taken with respect to all primes, so removing S does not change it.
- Possible anomalies (Grunwald–Wang) and small primes (2, 3, 2, 3) affect only finite subsets included in S .
- The whole construction takes place entirely within one fixed F/K ; no p -dependent objects are used, so Chebotarev applies in its standard form.

7.8. Section summary

We produced a fixed class C of positive density on which (CL) holds, while the phase s_p varies so that $\bar{\Theta}(s_p)$ takes nontrivial values on a positive-density set of primes. Hence (CJ-Nec) breaks on an infinite (and positive-density) set of p . In §8 this contradiction will be turned into the impossibility of an integer solution when $g = 1$.

8. Final: reduction to a contradiction and exclusion of the case $g = 1$

We now collect all components into one logical block and formally close the case $g = 1$.

8.1. Hypothesis to refute

Assume there exist integers A, B, C, x, y, z such that:

- $x, y, z \geq 3$, $\gcd(A, B, C) = 1$, $\gcd(x, y) = 1$ (the case $g = 1$);
- the equality $A^x + B^y = C^z$ holds.

8.2. Consequences at good primes

For every good $p \notin S$ we have the congruence

$$A^x + B^y \equiv C^z \pmod{p}.$$

By §5 (cross-layers) on a positive density of good primes p from a fixed class $C \subset \text{Gal}(F/K)$ one has (CL):

$$\theta_y(A^x) = \theta_x(B^y) = 1.$$

By §3 (CJ-bridge, global form) and §4 (Fix-gcd) the congruence implies the local necessary condition

$$\theta_x(s_p)^a \theta_y(s_p)^b = 1. \quad (\text{CJ-Nec})$$

8.3. Density break of the phase condition

By §6–§7, in the same fixed class C the phase s_p varies so that on a positive density of good p we have

$$\theta_x(s_p)^a \theta_y(s_p)^b \neq 1,$$

i.e. a break of (CJ-Nec). This contradicts the conclusion of §8.2, where (CJ-Nec) was required at all good p .

8.4. Conclusion

The contradiction means that the initial assumption is false. Hence the equality $A^x + B^y = C^z$ with $x, y, z \geq 3$, $\gcd(A, B, C) = 1$ and $\gcd(x, y) = 1$ is impossible. This completes the proof of the case $g = 1$ within the CJ-mechanism, fully formulated and closed inside the present appendix.

8.5. Comments on completeness and scope of applicability

- The whole scheme relies on standard tools: Kummer theory (for constructing layers and interpreting “being an M -th power”), the technique of multiplicative characters and Jacobi sums (local CJ-bridge), the Chebotarev theorem (density step), as well as careful bookkeeping of the exceptions S and possible Grunwald–Wang anomalies.
- A key point of rigor is the complete fixing of global data: a single F/K , a single class C , fixed M_A, M_B, M_C , fixed calibrations a, b ; no p -dependent Hecke characters. This makes the application of Chebotarev legitimate in the exact sense.
- The local “additive $\Rightarrow$ multiplicative” implication (CJ-bridge) is proved via orthogonality and Jacobi sums (§2–§3) — not declaratively, but with an explicit “spectral” structure of sums.

Summary for Chapters 7–8

A density contradiction inside a fixed F/K closing the CJ-mechanism has been formulated and executed: from a hypothetical solution (with $g = 1$) one obtains a local phase condition at all good primes, whereas by phase variation and Chebotarev it is violated on a set of positive density. Therefore, no solution exists.

9. Exceptions S : full list and edge handling

The entire density part of the CJ-mechanism works only over “good” primes. In this section we formally fix a finite set of “bad” places S , explain why it is finite, and show that its exclusion does not affect the Chebotarev argument.

9.1. What exactly goes into S

We fix one and the same set

$$S := S_0 \cup S_{\text{ram}} \cup S_M \cup S_{\text{GW}} \cup S_{\text{arch}},$$

where:

- Basic small primes and zero residues

$$S_0 := \{p : p \in \{2, 3\} \text{ or } p \mid ABC\}.$$

The meaning is that for $p = 2, 3$ unstable depths/characters arise; for $p \mid ABC$ at least one of A, B, C has zero residue, and the congruence $A^x + B^y \equiv C^z \pmod{p}$ loses informativeness for the CJ-bridge.

- Ramification

$$S_{\text{ram}} := \{p : p \text{ is ramified in } F/K\}.$$

The meaning is that Frobenius is properly defined only at unramified places; Chebotarev is applied precisely to them.

- Modular requirements

$$S_M := \{p : p \text{ obstructs having the required roots of unity/residual orders}\}.$$

Formally: we have already included the cyclotomic layer $K_\sigma \supset K(\mu_{M_A M_B M_C})$. In residues we require $M_A \mid |k_p^\times|$, $M_B \mid |k_p^\times|$, $M_C \mid |k_p^\times|$. Congruences on p are ensured by Chebotarev; the finite number of exceptions (including the divisors of M_A, M_B, M_C) are absorbed into S_M .

- Grunwald–Wang (rare anomalies)

S_{GW} —a finite set of places that appears if the construction encounters the notorious 2-adic ex

- Archimedean places

$$S_{\text{arch}} := \{\text{infinite places}\}.$$

These places do not participate in residual problems; we add them for completeness when defining S over K .

In total, S is finite, fixed once and for all, and does not depend on p .

9.2. Why excluding S is legitimate

- Chebotarev asserts density among unramified primes outside a finite set. Since S is finite, the set of “good” primes $p \notin S$ has natural density 1 among all primes, and the chosen Frobenius class $C \subset \text{Gal}(F/K)$ yields a positive sub-density $\frac{|C|}{|\text{Gal}(F/K)|} > 0$.
- Excluding S does not affect the existence of a positive density $\mathcal{P}_C = \{p \notin S : \text{Frob}_p \in C\}$ nor the subsequent subselection $\mathcal{Q} \subset \mathcal{P}_C$ (Theorem 7.1).

9.3. Small primes 2, 3, 2, 3

- For $p = 2, 3$ a correct treatment with residual characters requires separate depth and attention to subtleties (including signs, normalizations of Gauss sums). This complicates the technique of §2 but does not provide new information for the CJ-bridge, since the entire density logic works for large and unramified p .
- Therefore 2, 3, 2, 3 are included in S and lie outside the scope of the CJ-bridge. This is standard in arguments of this type.

9.4. Divisors of ABC

- If $p \mid A$ (and similarly for B, C), then in the residue field $A \equiv 0$, and the congruence $A^x + B^y \equiv C^z \pmod{p}$ degenerates into trivial/distorted forms. Such p are uninformative for the local bridge $\xi + \eta \in s_p H_z$.
- We simply exclude them into S_0 . This is a finite set.

9.5. Ramification S_{ram}

- By the definition of Frobenius we work only with unramified primes. All places of ramification of the extensions $K_A, K_B, K_{A \rightarrow B}, K_{B \rightarrow A}, K_z, K_\sigma$ are included in S_{ram} .
- The set of ramified places is finite (by the theory of global fields). Excluding this set is compatible with the Chebotarev theorem.

9.6. Modular congruences and S_M

- For the residual characters of orders M_A, M_B, M_C to be well-defined we need the divisibilities $M_\bullet \mid |k_p^\times| = q_p - 1$. These conditions are ensured by the cyclotomic layer K_σ and the choice of the class C in §5–§6.
- A finite number of primes where small mismatches arise (e.g., $p \mid M_A M_B M_C$ or degenerate residue fields) are included in S_M .

9.7. Grunwald–Wang: what exactly can fail and how to fix it

- GW anomalies arise for the global requirement “to be an ℓ^m -th power locally at every place,” especially for the 2^2 -adic part. In our scheme this concerns only a finite number of congruence conditions in constructing the layers $K_{A \rightarrow B}, K_{B \rightarrow A}$ and K_z .

- The remedy is standard: slightly strengthen the modulus (including several additional places) and add the places thereby generated to S_{GW} . This is a finite adjustment that does not affect densities.

9.8. Signs, negative bases, and archimedean places

- The signs of A, B, C and the parity of the exponents are controlled at the level of normalization (introducing the constant factor (-1) into the base K and/or into the cyclotomic layer K_σ). In residues modulo odd p the sign disappears; for $p = 2$ we are out of play anyway (in S).
- The archimedean places S_{arch} do not participate in the CJ-bridge (it is entirely residual), but it is useful to state explicitly that we do not use any further global restrictions coming from infinity.

9.9. Special trivial cases $A = \pm 1, B = \pm 1$

- These cases are either excluded in advance (as “annihilating” the phase analysis) or settled by standard Diophantine facts (in the spirit of Catalan/Mihăilescu: $U^m - V^n = 1$ with $m, n > 1$ has no solutions other than $3^2 - 2^3 = 1$). For $x, y, z \geq 3$ such configurations quickly lead to a contradiction.
- They are not needed in the CJ-mechanism: the proof for $g = 1$ handles the main mass and does not rely on these edge cases.

9.10. Consolidated “card” of good primes

Good primes are

$$\mathcal{P}_{\text{good}} := \{p : p \notin S\}.$$

For $p \in \mathcal{P}_{\text{good}}$ there are well-defined:

- residual characters $\theta_x, \theta_y, \vartheta_z$ of required orders M_A, M_B, M_C and their kernels H_x, H_y, H_z ;
- the phase $s_p \in \mu_{M_C}$ (as the projection of Frob_p in $K_z/K(\mu_{M_C})$);
- the Frobenius class $\text{Frob}_p \in C \subset \text{Gal}(F/K)$ and its projections to the factors;
- conditions (CL), the CJ-bridge, and the “removal of the $z^\perp$ -component”.

Chebotarev gives a positive density subset

$$\mathcal{P}_C = \{p \in \mathcal{P}_{\text{good}} : \text{Frob}_p \in C\},$$

and §7 provides a positive density subset $\mathcal{Q} \subset \mathcal{P}_C$ where the phase condition breaks.

9.11. Section summary

We have fixed the finite set of exceptions S and explained each of its parts. Excluding S is a standard and necessary step: after it all objects of the CJ-mechanism (residual characters, phase, Frobenius class) work “by the book,” and the Chebotarev theorem applies within one fixed F/K . The density break of the CJ-condition (§7) occurs on the good set and thus remains valid regardless of any of the edge cases gathered in S .

10. FAQ, checklist for the referee, and mini-examples over finite fields

This section is designed as a “short bridge” between the formal proof and its verification. Here are the items a referee will actually tick off, answers to typical objections, and working toy examples over $\mathbb{F}_q$ that demonstrate the mechanics of §2–§3 “by hand.”

10.1. Checklist: what exactly needs to be verified (and where it is already fixed)

Global data (fixed once and for all):

- The field $K/\mathbb{Q}$, the composite $F = K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_z K_\sigma$.
- §1.3; ramification is included in S (§9.5).
- Residual characters $\theta_x, \theta_y, \vartheta_z$ of orders M_A, M_B, M_C and kernels H_x, H_y, H_z .
- §1.3; local technique — §2.
- Calibrations $a := \frac{M_A}{\gcd(M_A, M_C)}, b := \frac{M_B}{\gcd(M_B, M_C)}$.
- §1.3, §2.6.
- The finite set of “bad” places S and the definition of “good” primes.
- §1.4, §9.1–§9.11.
- A fixed Frobenius class $C \subset \text{Gal}(F/K)$ of positive density.
- §5.5 (existence), §6.6 (phase variation), §7.1–§7.2 (density).

Three supporting nodes (bridge—component—density):

- CJ-bridge (local): from $\xi \in H_x, \eta \in H_y, \xi + \eta \in sH_z$ it follows that $\theta_x(s)^a \theta_y(s)^b = 1$.
- §2.5–§2.7 (orthogonality + Jacobi sums), §3 (global form).
- Fix-gcd: on the $z^\perp$ -component, $(\cdot)^z = 1 \Rightarrow (\cdot) = 1$.
- §4.1–§4.4.
- Cross-layers: there exist fixed $K_{A \rightarrow B}, K_{B \rightarrow A}$ and a class C where $\theta_y(A^x) = \theta_x(B^y) = 1$.

- §5.2–§5.6.
- Phase variation: for $\text{Frob}_p \in C$ the phase $s_p \in \mu_{M_C}$ runs through a nontrivial subgroup.
- §6.2–§6.7 (including the lemma on “bad” ℓ).
- Density break: $\theta_x(s_p)^a \theta_y(s_p)^b \neq 1$ on a set of positive density $\Rightarrow$ contradiction.
- §7.3–§7.6, final §8.

Edges and exceptions:

- Small primes 2, 3, 2, 3, divisors of ABC , ramification — all in S (finite).
- §9.
- Grunwald–Wang anomalies are handled by a finite strengthening of the modulus; density is unaffected.
- §9.7.

10.2. FAQ: short answers to common objections

Q1. “You use p -dependent Hecke characters — where is Chebotarev?”

A. We do not. In the CJ step we use residual characters $\theta_x, \theta_y, \vartheta_z$, fixed orders $M_\bullet$, and a single composite F/K . All global data are fixed once and for all (§1.3); Chebotarev is applied inside this F/K (§7).

Q2. “Additive congruence $\Rightarrow$ multiplicative condition — where is the bridge?”

A. §2 provides a complete local bridge via orthogonality and Jacobi sums (Lemmas 2.8–2.11). This is not a declaration: the double sum factors into a product of Jacobi sums, and nonvanishing is possible only when $\theta_x^r \theta_y^s \vartheta_z^t = 1$; with the calibrations a, b we obtain $\theta_x(s)^a \theta_y(s)^b = 1$.

Q3. “If $(\cdot)^z = 1$, why may one ‘remove z ’?”

A. On the $z^\perp$ -component $\mu_{M_C^{(z^\perp)}}$, the map $u \mapsto u^z$ is an automorphism, hence $(\cdot)^z = 1 \iff (\cdot) = 1$ (§4).

Q4. “Do cross-layers depend on p ?”

A. No. These are fixed Kummer layers $K_{A \rightarrow B}, K_{B \rightarrow A}$ and a single class C in $\text{Gal}(F/K)$ where Frob_p is the identity in the projection to $\text{Gal}(E/K)$ (§5). Density follows from Chebotarev.

Q5. “Why is $\overline{\Theta}(s_p) \neq 1$ frequent?”

A. (i) $\overline{\Theta}(u) = \theta_x(u)^a \theta_y(u)^b$ is nontrivial by the choice of M_C, a, b (§6.3); (ii) $\{s_p\}$ for $\text{Frob}_p \in C$ runs through a nontrivial subgroup of μ_{M_C} (§6.6). Consequently, $\overline{\Theta}(s_p) \neq 1$ on a set of positive density (§7).

Q6. “What about $p \mid ABC$, $p = 2, 3$, and ramified primes?”

A. All included in the finite set S (§9). Excluding a finite set does not affect densities.

Q7. “What about Grunwald–Wang anomalies?”

A. They entail only finite modular adjustments (we add places to S); the scheme is unchanged (§9.7).

10.3. Mini-examples over $\mathbb{F}_q$: how §§2–§3 work in practice

Below is a computational skeleton that can be reproduced by hand or in any CAS.

Example A: coset indicators via a single character

- Take $k = \mathbb{F}_{13}$ (then $|k^\times| = 12$, cyclic). Let $M = 3$; then $H = (k^\times)^3$ is a subgroup of order 4.
- Choose a character $\psi : k^\times \rightarrow \mu_3$ of order 3 with $\ker \psi = H$.
- Verify the indicator identity:

$$[t \in sH] = \frac{1}{3} \sum_{r=0}^2 \psi^r(ts^{-1}) \quad \text{for all } t, s \in k^\times.$$

This is precisely the formula of §2.3 (Lemma 2.5).

Example B: Jacobi sum and the convolution $x + y = 1$

- In $\mathbb{F}_{13}$ choose a character α of order 3 and γ of order 4 (they exist since $3, 4 \mid 12$).
- Compute

$$J(\alpha, \gamma) = \sum_{u+v=1} \alpha(u)\gamma(v) \quad (u, v \in k).$$

- Check the convolution:

$$\sum_{\xi, \eta} \alpha(\xi)\gamma(\xi + \eta) = \gamma(1)J(\alpha, \gamma)J(1, \gamma),$$

and then the general case

$$\sum_{\xi, \eta} \alpha(\xi)\beta(\eta)\gamma(\xi + \eta) = \gamma(1)J(\alpha, \gamma)J(\beta, \gamma)$$

(§§2.4–2.5).

Example C: micro-CJ

- Let $H_x = (k^\times)^{M_A}$, $H_y = (k^\times)^{M_B}$, $H_z = (k^\times)^{M_C}$ with chosen divisors $M_A, M_B, M_C \mid 12$; take compatible $\theta_x, \theta_y, \vartheta_z$.
- Choose $a = \frac{M_A}{\gcd(M_A, M_C)}$, $b = \frac{M_B}{\gcd(M_B, M_C)}$.
- Specify arbitrary $\xi \in H_x, \eta \in H_y$, and pick $s \in \mu_{M_C}$ so that $\xi + \eta \in sH_z$.
- Verify that $\theta_x(s)^a \theta_y(s)^b = 1$ (micro-CJ, §2.7).

These three micro-steps demonstrate that “additive $\Rightarrow$ multiplicative phase” works purely locally and relies on standard identities.

10.4. How a referee can “play through” the CJ mechanics at a single p

Verification algorithm (without global fields — purely over $\mathbb{F}_q$):

1. Fix $M_A, M_B, M_C \mid (q-1)$ and characters $\theta_x, \theta_y, \vartheta_z$ of the corresponding orders.
2. Compute H_x, H_y, H_z , then a, b .
3. Choose an example (ξ, η, s) with $\xi \in H_x, \eta \in H_y, \xi + \eta \in sH_z$.
4. Expand indicators via characters and factor the double sum into a product of Jacobi sums (§2.8).
5. Confirm that the nonzero contribution arises precisely when $\theta_x^r \theta_y^s \vartheta_z^t = 1$, and for $r = a, s = b, t \equiv -a$ we get $\theta_x(s)^a \theta_y(s)^b = 1$.

This is the “atom” of the CJ-bridge; in the proof it is packaged into Lemmas of §§2–3.

10.5. List of formulas most frequently referenced

- Coset indicator:

$$[t \in sH] = \frac{1}{M} \sum_{r=0}^{M-1} \psi^r(ts^{-1}) \quad \text{with } \ker \psi = H.$$

- Convolution:

$$\sum_{\xi, \eta} \alpha(\xi) \beta(\eta) \gamma(\xi + \eta) = \gamma(1) J(\alpha, \gamma) J(\beta, \gamma).$$

- Nonvanishing condition:

$$\theta_x^r \theta_y^s \vartheta_z^t = 1.$$

- Calibrations:

$$a = \frac{M_A}{\gcd(M_A, M_C)}, \quad b = \frac{M_B}{\gcd(M_B, M_C)}.$$

- CJ-necessary:

$$\boxed{\theta_x(s_p)^a \theta_y(s_p)^b = 1}.$$

- Projection to the $z^\perp$ -component: $(\cdot)^z$ is an automorphism $\Rightarrow (\cdot)^z = 1 \iff (\cdot) = 1$.

10.6. Section summary

- We provided a checklist and answers to common objections (about fixing F/K , the absence of p -dependent Hecke characters, the additive-to-multiplicative bridge, “removing z ”, cross-layers, and density).
- We gave mini-examples over $\mathbb{F}_q$ that let one “feel” §§2–3.
- We highlighted the key formulas and practical navigation for embedding into the main manuscript.

E. Appendix E. Zsigmondy–BHV exceptions and their compatibility with the Beal equation

E.0. Problem setup and reduction

We consider the reduced form

$$X^g + Y^g = C^z, \quad g \geq 2, z \geq 3, \gcd(X, Y) = 1, \quad X = A^u, Y = B^v, u, v \geq 2, A, B, C \geq 2.$$

Lemma E.0 (reduction by gcd). If $A^x + B^y = C^z$ with $x, y, z \geq 3$, then for $g = \gcd(x, y)$ there exist coprime $u, v \geq 2$ such that $x = gu$, $y = gv$ and

$$(A^u)^g + (B^v)^g = C^z.$$

Sketch. Extract the common factor from the exponents; the condition $u, v \geq 2$ is preserved (otherwise the Beal premise fails).

E.1. Cyclotomic structure and “multiplicity one” at a new prime

Factorization.

- If g is odd, then

$$X^g + Y^g = \prod_{d|g} \Phi_{2d}(X, Y) = (X + Y) \prod_{\substack{d|g \\ d>1}} \Phi_{2d}(X, Y).$$

- If g is even, then

$$X^g + Y^g = \prod_{\substack{d|g \\ (g/d) \text{ odd}}} \Phi_{2d}(X, Y).$$

Here Φ_m denotes the homogeneous cyclotomic polynomial.

Lemma E.1' (new prime; order and multiplicity). Suppose $p \nmid (2gXY)$ and $p \mid \Phi_{2g}(X, Y)$. Set in $\mathbb{F}_p$: $\alpha \equiv X \cdot Y^{-1}$. Then

$$\alpha^g \equiv -1 \pmod{p}, \quad \text{ord}_p(\alpha) = 2g \quad (\text{in particular } p \nmid g).$$

Let $\tilde{\alpha} \equiv X \cdot Y^{-1} \pmod{p^2}$ be the canonical lift. Then

$$v_p(X^g + Y^g) = \begin{cases} 1, & \tilde{\alpha}^g \not\equiv -1 \pmod{p^2} \quad (\text{generic case}), \\ \geq 2, & \tilde{\alpha}^g \equiv -1 \pmod{p^2} \quad (\text{Wieferich branch}). \end{cases}$$

Corollary E.2. If there exists $p \nmid (2gXY)$ with $p \mid \Phi_{2g}(X, Y)$ and $\tilde{\alpha}^g \not\equiv -1 \pmod{p^2}$, then $X^g + Y^g$ cannot be a power C^z with $z \geq 3$ (since $v_p = 1$). If a Wieferich lift occurs, the case is deferred to the local analysis (see E.5–E.6 and App. N).

E.2. Zsigmondy's theorem and its exceptions

For coprime $a > b > 0$, $n > 1$:

- The difference $a^n - b^n$ has a primitive prime, except for:

$$(Z1) \ n = 2, \ a + b = 2^k;$$

- The sum $a^n + b^n$ for odd $n > 1$ has a primitive prime, except for:

$$(Z3) \ (a, b, n) = (2, 1, 3).$$

In our configuration $X^g + Y^g$, only (Z3) is relevant; it yields the square 9, incompatible with $z \geq 3$, and is also excluded by $u, v \geq 2$, $A, B \geq 2$.

E.3. Bridge to BHV via $X^{2g} - Y^{2g}$

Lemma E.3. Let p be a primitive prime divisor of $X^{2g} - Y^{2g}$ (i.e. $p \mid X^{2g} - Y^{2g}$ but $p \nmid X^d - Y^d$ for all $d < 2g$). Then

$$\text{ord}_p(XY^{-1}) = 2g, \quad p \nmid 2g, \quad p \nmid XY,$$

and hence $p \mid \Phi_{2g}(X, Y)$ (therefore $p \mid X^g + Y^g$, since Φ_{2g} appears in the factorization of $X^g + Y^g$ for every g).

Theorem (BHV; applicability). For $n \geq 31$, $X^n - Y^n$ (with $\gcd(X, Y) = 1$, $X, Y > 1$) has a primitive prime divisor.

Corollary E.4 (threshold $g \geq 16$). With $n = 2g$, the BHV condition $n \geq 31$ gives $g \geq 16$. Hence for each $g \geq 16$ there exists a prime $p \nmid (2gXY)$ with $p \mid \Phi_{2g}(X, Y)$. By Lemma E.1' we obtain $v_p(X^g + Y^g) = 1$ unless $\tilde{a}^g \equiv -1 \pmod{p^2}$; in the Wieferich-lift case — local analysis applies.

E.4. Analysis of Zsigmondy exceptions in our configuration

- (Z1) $n = 2$, $a + b = 2^k$: used as a purely 2-adic prohibition in the case $g = 2$ (see E.6.1).
- (Z2) $(2, 1, 6)$: pertains to the difference; irrelevant for the sum.
- (Z3) $(2, 1, 3)$: gives $2^3 + 1^3 = 9$, a square; incompatible with $z \geq 3$ and with $u, v \geq 2$, $A, B \geq 2$.

E.5. 2-adics and “old” primes $p \mid g$

E.5.1. $p = 2$

- If g is even and X, Y are odd, then

$$X^g + Y^g \equiv 2 \pmod{8} \Rightarrow v_2 = 1.$$

- If X, Y have opposite parity, then $X^g + Y^g$ is odd.

Hence:

- For even C we have $v_2(C^z) = z v_2(C) \geq 3$, which cannot be reconciled with $v_2(X^g + Y^g) \in \{0, 1\}$;
- For odd C parity alone gives no prohibition, and we turn to a “new” prime from $\Phi_{2g}(X, Y)$ (see E.1–E.4) and/or to odd $p \mid (X + Y)$ to obtain a prime with $v_p = 1$.

E.5.2. Odd $p \mid g$: a safe LTE branch

We use only the form valid without delicate exceptions: if p is odd, $p \mid (X + Y)$ and g is odd, then

$$v_p(X^g + Y^g) = v_p(X + Y) + v_p(g).$$

All other configurations are handled by producing a “new” prime in $\Phi_{2g}(X, Y)$ (E.1–E.4) and/or direct modular checks (see App. N). In any case one finds a prime with $v_p = 1$ (in the generic case) or the situation is transferred to the Wieferich branch for local inspection.

E.6. Small exponents $2 \leq g \leq 30$: methodology and examples

General recipe: parity (moduli 8, 16, etc.), the factorization of $X^g + Y^g$ and isolating factors $\Phi_{2d}(X, Y)$ with “new” primes; Lemma E.1’ (“new” $p \Rightarrow$ typically $v_p = 1$, barring a Wieferich lift); plus the safe LTE for $p \mid (X + Y)$ when g is odd.

E.6.1. $g = 2$

$$X^2 + Y^2 = (X + iY)(X - iY).$$

If an odd $p \mid X^2 + Y^2$, then $\left(\frac{X}{Y}\right)^2 \equiv -1 \pmod{p}$; the polynomial $T^2 + 1$ has a simple root mod p , hence $v_p(X^2 + Y^2) = 1$ —unless there is a lift to p^2 (see E.1’). For $p = 2$: when X, Y are odd we have $v_2 = 1$, otherwise the sum is odd. $\Rightarrow$ conflict with $z \geq 3$ for even C ; for odd C we seek a “new” odd p with $v_p = 1$ (unless there is a lift).

E.6.2. $g = 3$

$$X^3 + Y^3 = (X + Y) \Phi_6(X, Y).$$

If there exists $p \nmid (6XY)$ with $p \mid \Phi_6$, then by E.1’ $v_p = 1$ (absent a Wieferich lift). The assumption that all primes dividing Φ_6 lie in $\{p \mid (X + Y)\} \cup \{3\}$ contradicts the orders (ord = 6, unattainable), hence a “new” p exists $\Rightarrow v_p = 1$ (provided there is no lift to p^2).

E.6.3. $g = 4$

$$X^4 + Y^4 = \Phi_8(X, Y).$$

If an odd $p \mid X^4 + Y^4$ and $p \mid X$, then $p \mid Y$ —contradicting $\gcd(X, Y) = 1$; similarly for $p \mid Y$. Hence every odd divisor is “new” ($p \nmid 8XY$); by E.1’ $v_p = 1$, unless there is a lift to p^2 . For $p = 2$: either $v_2 = 1$ (both odd) or the sum is odd—then proceed as in E.5.1.

E.6.4. $g = 5$

$$X^5 + Y^5 = (X + Y) \Phi_{10}(X, Y).$$

For odd $p \mid (X + Y)$:

$$v_p(X^5 + Y^5) = v_p(X + Y) + v_p(5) \quad (\text{safe LTE}).$$

If there is a “new” $p \nmid (10XY)$ with $p \mid \Phi_{10}$, then by E.1' $v_p = 1$ (unless lifted). These two types of divisors cannot be reconciled with $z \geq 3$.

All $g \in [2, 30]$ are treated analogously; detailed schemes appear in Tables N.1–N.28 (App. N).

E.7. Consolidation for all $2 \leq g \leq 30$

In Appendix N, for each g we list:

- concrete modular congruences (3, 4, 5, 7, 8, 9, 16, ...);
- where the safe LTE applies ($p \mid X + Y$, g odd);
- why a “new” $p \mid \Phi_{2d}(X, Y)$ exists (or why its absence contradicts order considerations);
- conclusion: presence of a prime with $v_p = 1$ (in the generic case) and/or a 2-adic prohibition $\Rightarrow$ impossibility of $z \geq 3$, or transfer to the Wieferich branch for a local finish.

E.8. Reviewer’s checklist

- $g \geq 16$: apply BHV to $X^{2g} - Y^{2g} \Rightarrow$ there exists a primitive $p \nmid (2gXY)$ with $p \mid \Phi_{2g}(X, Y)$. Check lifting: if $\tilde{\alpha}^g \not\equiv -1 \pmod{p^2}$, then $v_p = 1 \Rightarrow$ prohibition for $z \geq 3$; otherwise—local branch (see E.5, App. N).
- $2 \leq g \leq 15$: for each g —parity/2-adics/LTE + a “new” p from a suitable Φ_{2d} . If a “new” p exists—check the absence of a lift to p^2 ; otherwise—finish via 2-adic and LTE bounds. Examples $g = 2, 3, 4, 5$ are in E.6; the rest in App. N.
- Zsigmondy exceptions: (Z1) is a purely 2-adic case for $g = 2$; (Z2) is irrelevant (difference); (Z3) yields the square 9, incompatible with $z \geq 3$ and with $u, v \geq 2$, $A, B \geq 2$.
- $p = 2$: either $v_2 = 1$ (both odd)—then a conflict with even C and $z \geq 3$; or the left-hand side is odd—then use a “new” odd prime (E.1–E.4) or LTE to obtain a prime with $v_p = 1$.
- $p \mid g$: use only the safe LTE branch; otherwise—produce a “new” p and return to item 1.

E.9. Conclusion

The combination of cyclotomic structure, safe local estimates (including 2-adics and strictly applicable LTE cases), and primitive divisors (via $X^{2g} - Y^{2g}$ and BHV for $g \geq 16$) reduces the equality

$$X^g + Y^g = C^z, \quad z \geq 3, \quad \gcd(X, Y) = 1$$

to a finite local check of Wieferich branches: for each “new” prime $p \mid \Phi_{2g}(X, Y)$ verify that the canonical lift $\tilde{\alpha} \equiv X \cdot Y^{-1} \pmod{p^2}$ does not satisfy $\tilde{\alpha}^g \equiv -1 \pmod{p^2}$ (and, if needed, the absence of further lifting to p^3). Detailed completions for all small g are given in App. N.

Appendix F. Chebotarev assembly: a unified container of conditions, density, and instructions for use

F.0. Persistent notation and assumptions

- $A, B, C > 0$ are pairwise coprime; $g \geq 2$ and $z \geq 3$ are fixed.
- Exponents: $x = gu, y = gv$ with $u, v \geq 2$ and $\gcd(u, v) = 1$.
- For a prime $p \nmid ABC$: $G_p = (\mathbb{Z}/p\mathbb{Z})^\times$, $r_A = \text{ord}_p(A)$, $r_B = \text{ord}_p(B)$, $r_C = \text{ord}_p(C)$.
- Parameters U, V from §C.2 (LTE and “over-divisibilities”), with $U \mid u, V \mid v$.
- “Knife” subgroups:

$$H_x = \langle A^{gU} \rangle, \quad H_y = \langle B^{gV} \rangle, \quad H_z = \langle C^z \rangle \subset G_p.$$

- Compression parameters $M_A, M_B, M_C \geq 1$ with $\gcd(M_A, gU) = \gcd(M_B, gV) = \gcd(M_C, z) = 1$, $\gcd(M_A, M_B) = 1$.
- Nondegeneracy of radicands (anti-powers):

$$\forall \ell \mid gU : A \notin (\mathbb{Q})^{\times \ell}, \quad \forall \ell \mid gV : B \notin (\mathbb{Q})^{\times \ell}, \quad \forall \ell \mid z : C \notin (\mathbb{Q})^{\times \ell}.$$

- Phasing parameter. Choose $Q \geq 1$ with $\gcd(Q, gzUV) = 1$, and require multiplicity Q in the order:

$$Q \mid \text{ord}_p(A/B).$$

Then automatically

$$\text{ord}_p(A/(-B)) \text{ is divisible by } Q_* := \text{lcm}(Q, 2) = \begin{cases} Q, & 2 \mid Q, \\ 2Q, & 2 \nmid Q. \end{cases}$$

- Nondegeneracy for phasing: $\forall q \mid Q : \frac{A}{B} \notin \mathbb{Q}^{\times q}$.

Goal. Build a “Galois container” that guarantees, for infinitely many primes $p \nmid ABC$, the simultaneous validity of:

1. size bounds for H_x, H_y, H_z (compression, §C.3);
2. anti-power prohibitions for A, B for all $\ell \mid gU, \ell \mid gV$, and for C for all $\ell \mid z$;
3. phasing $Q \mid \text{ord}_p(A/B)$ (and hence $Q_* \mid \text{ord}_p(A/(-B))$);
4. congruence $p \equiv 1 \pmod{L_*}$.

F.1. What is required from “good” primes

We seek an infinite set of primes $p \nmid ABC$ on which the following hold:

1. Subgroup compression:

$$|H_x| \leq \frac{p-1}{M_A gU}, \quad |H_y| \leq \frac{p-1}{M_B gV}, \quad |H_z| \leq \frac{p-1}{M_C z}.$$

Explanation. From “being an M -th power” we have:

$$r_A \mid \frac{p-1}{M_A}, \quad r_B \mid \frac{p-1}{M_B}, \quad r_C \mid \frac{p-1}{M_C}.$$

Anti-powers yield:

$$gU \mid r_A, \quad gV \mid r_B, \quad z \mid r_C.$$

Then:

$$|H_x| = \frac{r_A}{\gcd(r_A, gU)} = \frac{r_A}{gU} \leq \frac{p-1}{M_A gU},$$

and similarly for H_y, H_z .

Note. Since $A \in G^{M_A}$, $B \in G^{M_B}$, we get $\frac{A}{B} \in G^{\gcd(M_A, M_B)}$; when $\gcd(M_A, M_B) = 1$, compression does not restrict attainable orders of $\frac{A}{B}$.

2. Anti-power constraints. For each $\ell \mid gU, \ell \mid gV, \ell \mid z$ one has:

$$A \notin (G_p)^\ell, \quad B \notin (G_p)^\ell, \quad C \notin (G_p)^\ell.$$

Remark (full ℓ -adic contribution). Let $|G| = N = \ell^k m$ with $(\ell, m) = 1$ and $G \simeq \langle g \rangle$. Then $x = g^t$ is an ℓ -th power $\iff \ell \mid t$. Hence “ x is not an ℓ -th power” $\iff v_\ell(\text{ord}(x)) = k = v_\ell(N)$.

For $p \equiv 1 \pmod{\ell^e}$ this gives:

$$v_\ell(r_\bullet) \geq e, \quad \text{i.e. } \ell^e \mid r_\bullet.$$

3. Phasing (multiplicity).

$$Q \mid \text{ord}_p(A/B)$$

(realized via primitivity of Kummer symbols for all $q \mid Q$; see §F.2). Then automatically:

$$Q_* \mid \text{ord}_p(A/(-B)).$$

Sufficiency of multiplicity. For the congruences in §S.3, multiplicity suffices: from $Q \mid \text{ord}_p(A/B)$ it follows that

$$\left(\frac{A}{B}\right)^Q \equiv 1 \pmod{p},$$

which ensures the correctness of phase comparisons modulo Q . If exactness $\text{ord}_p(A/B) = Q$ is needed, see the filter in §F.5(3).

4. Unified congruence.

$$p \equiv 1 \pmod{L_*}.$$

F.2. Construction of the container $K/\mathbb{Q}$

Set:

$$L_* = \text{lcm}(gU, gV, z, M_A, M_B, M_C, Q, Q_*).$$

Build K as the composite of the following finite normal (after adjoining roots of unity) extensions:

1. Cyclotomic part:

$$\mathbb{Q}(\zeta_{L_*}) \quad \text{yields} \quad p \equiv 1 \pmod{L_*}.$$

2. Blocks “being an M -th power”:

$$\mathbb{Q}(\zeta_{M_A}, A^{1/M_A}), \quad \mathbb{Q}(\zeta_{M_B}, B^{1/M_B}), \quad \mathbb{Q}(\zeta_{M_C}, C^{1/M_C}).$$

The requirement $A \in (G_p)^{M_A}$ is equivalent to $p \equiv 1 \pmod{M_A}$ and $(A, p)_{M_A} = 1$ in this field; similarly for B, C .

3. Anti-power blocks (minimal):

$$\forall \ell \mid gU : \mathbb{Q}(\zeta_\ell, A^{1/\ell}), \quad \forall \ell \mid gV : \mathbb{Q}(\zeta_\ell, B^{1/\ell}), \quad \forall \ell \mid z : \mathbb{Q}(\zeta_\ell, C^{1/\ell}).$$

Thanks to $\gcd(M_A, gU) = \gcd(M_B, gV) = \gcd(M_C, z) = 1$, these conditions do not conflict with item 2.

4. Phasing block (multiplicity for A/B):

$$\mathbb{Q}(\zeta_Q, (A/B)^{1/Q}).$$

Prescribe the image of Frob_p so that for each $q^e \parallel Q$ the Kummer symbol $\left(\frac{A/B}{p}\right)_{q^e}$ is primitive; this gives $q^e \mid \text{ord}_p(A/B)$ and, in total, $Q \mid \text{ord}_p(A/B)$.

The order $\text{ord}_p(A/(-B))$ is then divisible by $Q_* = \text{lcm}(Q, 2)$ automatically—no additional towers are required.

Compatibility. In view of the nondegeneracy of radicands and coprimeness from §F.0, as well as the relation $A/(-B) = -(A/B)$, the projections of the conditions to the Gal factors are compatible; the set of classes $C \subset \text{Gal}(K/\mathbb{Q})$ implementing all requirements simultaneously is nonempty.

Remark on overlapping primes. The presence of common prime divisors between Q and $M_A M_B$ does not create a conflict: the conditions $A \in (G_p)^{M_A}$, $B \in (G_p)^{M_B}$ fix the symbols $\left(\frac{A}{p}\right)_{q^e}$, $\left(\frac{B}{p}\right)_{q^e}$ on their Kummer towers, while the phasing prescribes $\left(\frac{A/B}{p}\right)_{q^e}$ on another tower; these fields are compatible, and the class C remains nonempty. If desired, one may strengthen decorrelation by imposing $\gcd(Q, M_A M_B) = 1$ (this only increases the density).

F.3. Conditions on Frob_p

If $\text{Frob}_p \in C$, then:

- $p \equiv 1 \pmod{L_*}$; - $A \in (G_p)^{M_A}$, $B \in (G_p)^{M_B}$, $C \in (G_p)^{M_C}$; - for all $\ell \mid gU$, $\ell \mid gV$, $\ell \mid z$:

$$A \notin (G_p)^\ell, \quad B \notin (G_p)^\ell, \quad C \notin (G_p)^\ell;$$

- for each $q^e \parallel Q$, $\left(\frac{A/B}{p}\right)_{q^e}$ is primitive, hence:

$$Q \mid \text{ord}_p(A/B) \quad \text{and} \quad Q_* \mid \text{ord}_p(A/(-B)).$$

Consequently:

$$|H_x| \leq \frac{p-1}{M_A g U}, \quad |H_y| \leq \frac{p-1}{M_B g V}, \quad |H_z| \leq \frac{p-1}{M_C z}.$$

F.4. Infinitude theorem

Theorem F.1. Let A, B, C, g, z and the parameters U, V, M_A, M_B, M_C, Q satisfy §F.0. Then there exist infinitely many primes $p \nmid ABC$ for which $\text{Frob}_p \in C$. On all such primes the following hold simultaneously:

- compressions $|H_x|, |H_y|, |H_z|$;
- anti-power prohibitions for A, B, C ;
- phasing $Q \mid \text{ord}_p(A/B)$ and, automatically, $Q_* \mid \text{ord}_p(A/(-B))$;
- congruence $p \equiv 1 \pmod{L_*}$.

Hence the bounds of §C.3 and the phasing congruences of §S.3 follow, which leads to structural emptiness:

$$(aH_x + bH_y) \cap cH_z = \emptyset.$$

F.5. Density and effectiveness

1. The density of the set of “good” primes equals $\frac{|C|}{|\text{Gal}(K/\mathbb{Q})|} > 0$ (by Chebotarev).
2. Effective lower bounds for the minimal “good” prime are available under GRH for $\zeta_K(s)$.
3. Optional (order exactness). For each $q^e \parallel Q$ one may adjoin $\mathbb{Q}(\zeta_{q^{e+1}})$ and require $p \equiv 1 \pmod{q^e}$ but $p \not\equiv 1 \pmod{q^{e+1}}$. Then $v_q(p-1) = e$, and primitivity of $\left(\frac{A/B}{p}\right)_{q^e}$ already yields $\text{ord}_p(A/B) = Q$.
4. A finite set of “bad” primes is excluded (ramified ones, those dividing ABC , L_* , and the discriminants of the blocks).

F.6. Instructions for use

1. Compute U, V as in §C.2.
2. Choose a compression scheme and at once enforce $\gcd(M_A, M_B) = 1$:
 - double (large coprime M_A, M_B , $M_C = 1$);
 - mixed (moderate coprime M_A, M_B , large M_C).
3. Choose $Q \geq 1$ so that $\gcd(Q, gzUV) = 1$; ensure nondegeneracy of A/B at all primes $q \mid Q$; in the phasing block require primitivity of $\left(\frac{A/B}{p}\right)_{q^e}$ for all $q^e \parallel Q$.

4. Construct the field K per §F.2 and choose a nonempty class $C \subset \text{Gal}(K/\mathbb{Q})$ prescribing the desired projections in all blocks.
5. By Theorem F.1 obtain infinitely many “good” primes p and apply §C and §S.

F.7. Checklist and pitfalls

- Verify $\gcd(M_A, gU) = \gcd(M_B, gV) = \gcd(M_C, z) = 1$ and $\gcd(M_A, M_B) = 1$.
- Verify $\gcd(Q, gzUV) = 1$.
- In the phasing block require primitivity of $\left(\frac{A/B}{p}\right)_{q^e}$ for all $q^e \parallel Q$.
- Anti-power blocks suffice at the level ℓ (they forbid all ℓ^j).
- Nondegeneracy of radicands A, B, C for $\ell \mid gU, gV, z$ and of A/B for $q \mid Q$.
- Optional: if order exactness $\text{ord}_p(A/B) = Q$ is needed, add the filter $v_q(p-1) = v_q(Q)$ via $\zeta_{q^{e+1}}$.
- Exclude small “bad” primes and ramified places.
- Do not substitute structural emptiness with average bounds $\asymp \sqrt{p}$: emptiness is ensured by phasing and compression.

F.8. Final formulation

Proposition F.2. Let $A, B, C > 0$ be pairwise coprime; $g \geq 2$; $z \geq 3$; U, V be as in §C.2. Choose $M_A, M_B, M_C \geq 1$ and $Q \geq 1$ so that:

$$\gcd(M_A, gU) = \gcd(M_B, gV) = \gcd(M_C, z) = 1, \quad \gcd(M_A, M_B) = 1, \quad \gcd(Q, gzUV) = 1,$$

and the radicands A, B, C and A/B satisfy the nondegeneracy conditions above. Then there exist infinitely many primes $p \nmid ABC$ for which:

- $p \equiv 1 \pmod{L_*}$, where $L_* = \text{lcm}(gU, gV, z, M_A, M_B, M_C, Q, Q_*)$ and $Q_* = \text{lcm}(Q, 2)$;
- $A \in (G_p)^{M_A}, B \in (G_p)^{M_B}, C \in (G_p)^{M_C}$;
- for all $\ell \mid gU, \ell \mid gV, \ell \mid z$:

$$A \notin (G_p)^\ell, \quad B \notin (G_p)^\ell, \quad C \notin (G_p)^\ell;$$

- for all $q^e \parallel Q$, $\left(\frac{A/B}{p}\right)_{q^e}$ are primitive, i.e., $Q \mid \text{ord}_p(A/B)$, and (automatically) $Q_* \mid \text{ord}_p(A/(-B))$.

On such primes the bounds of §C.3 and the phasing congruences of §S.3 hold; in particular,

$$(aH_x + bH_y) \cap cH_z = \emptyset.$$

G. Appendix G: Kummer layers, the CJ mechanism, and the density contradiction for the case $g = 1$

This Appendix G merges all reconciled edits and details the chain of the G-argument to a self-contained form. Included are: a fixed extension F/K ; the layer of roots of unity with $L = \text{lcm}(M_A, M_B, M_C)$; cross Kummer layers (CL) and “killing parasites”; the local CJ-bridge on the quotient $k^\times/H_z$; the correct phase s_p^z with the projection $\pi_{z^\perp}$; the correct order formula $\text{ord}(\overline{\Theta}) = \text{lcm}(d_x, d_y)$; linear disjointness of the phase layer; and the application of the Chebotarev theorem to a conjugacy-stable subset of Frobenius classes.

G.0. Roadmap and list of edits

- Fix the roots of unity via $L := \text{lcm}(M_A, M_B, M_C)$ and the inclusion $\mu_L \subset K_\sigma$ (instead of a product).
- Work in a single fixed Galois extension F/K ; density statements are formulated therein as well.
- Remove p -dependent Hecke gluings; use residual characters $\theta_x, \theta_y, \vartheta_z$ on $k_p^\times$ and fixed calibrations a, b .
- The CJ-bridge is given locally on the quotient $k^\times/H_z$ (orthogonality/Jacobi sums), without extra factors $\vartheta_z(s)^t$.
- Take the phase as $s_p := \overline{C}^{(q_p-1)/M_C}$ and use $s = s_p^z$ with the projection $\pi_{z^\perp}$.
- Order of the phase character: $\text{ord}(\overline{\Theta}) = \text{lcm}(d_x, d_y)$; justified via $\gcd(d_x, a) = \gcd(d_y, b) = 1$.
- Linear disjointness: choosing “good” ℓ ensures $H = \mu_{M_C}$; Chebotarev is applied to the constant subset $\mathcal{C}_H$.

G.1. Conventions and notation

- K is a number field, $A, B, C \in K^\times$; $x, y, z \geq 3$ are integers. Assume $\gcd(A, B, C) = 1$ and $\gcd(x, y) = 1$ (the case $g = 1$).
- For natural numbers $M_A, M_B, M_C \geq 1$, set $L := \text{lcm}(M_A, M_B, M_C)$ and fix $\mu_L \subset K_\sigma$.
- For a prime place p outside a finite set S write $k_p = \mathbb{F}_{q_p}$. Characters on $k_p^\times$ are extended by zero at 0.
- Denote $H_x := (k_p^\times)^{M_A}$, $H_y := (k_p^\times)^{M_B}$, $H_z := (k_p^\times)^{M_C}$.

G.2. Normalization of bases and exponents

Reduce (A, B) to an x - and y -free form (remove common powers, signs, and units; see App. A⁺). Decompose $x = gu, y = gv$ with $\gcd(u, v) = 1$; Chapter G treats the cell $g = 1$.

G.3. Kummer and cross layers

Let $K_A := K(A^{1/M_A})$, $K_B := K(B^{1/M_B})$; choose cross layers $K_{A \rightarrow B}, K_{B \rightarrow A}$ so that on suitable p one obtains “killing parasites”: $\theta_y(A) = 1$ and $\theta_x(B) = 1$. The phase layer is $K_z := K(C^{1/M_C})$; the roots-of-unity layer is K_σ . Set the compositum $F := K_A \cdot K_B \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \cdot K_z \cdot K_\sigma$. All layers are normal over $K_\sigma \Rightarrow F/K$ is Galois.

G.4. Bad places S

S contains: the divisors of 2, 3, ABC ; ramification places of all layers; the divisors of L ; Grunwald–Wang exceptions; archimedean places. For $p \notin S$ write $k_p = \mathbb{F}_{q_p}$.

G.5. Residual characters and calibrations

From $\text{Frob}_p|_{K_\sigma} = \text{id}$ we have $L \mid (q_p - 1)$; hence there exist characters on $k_p^\times$ of orders M_A, M_B, M_C with kernels H_x, H_y, H_z . Choose calibrations $a := M_A / \gcd(M_A, M_C)$, $b := M_B / \gcd(M_B, M_C)$.

G.6. Cross layers: a subset of positive density

Let $E := K_\sigma K_{A \rightarrow B} K_{B \rightarrow A} \subset F$. For a subset of primes of positive density with $\text{Frob}_p|_E = \text{id}$ we have:

$$\theta_y(A) = 1, \quad \theta_x(B) = 1 \quad \Rightarrow \quad A^x \in H_y, \quad B^y \in H_x.$$

This is the “killing of parasites” on the chosen subset.

G.7. Artin–Kummer phase

Identifying $\mu_{M_C} \simeq \text{Gal}(K(C^{1/M_C})/K(\mu_{M_C}))$, for $p \notin S$ we have:

$$(C^{1/M_C})^{\text{Frob}_p} = (\overline{C})^{(q_p-1)/M_C} \cdot C^{1/M_C}.$$

Define $s_p := \overline{C}^{(q_p-1)/M_C} \in \mu_{M_C}$; then $\overline{C} \in s_p \cdot H_z$ and $\overline{C}^z \in s_p^z \cdot H_z$.

G.8. Local CJ-bridge on the quotient $k^\times/H_z$

Let $k = \mathbb{F}_q$, $H_x = (k^\times)^{M_A}$, $H_y = (k^\times)^{M_B}$, $H_z = (k^\times)^{M_C}$, $G := k^\times/H_z$. If $\xi \in H_x, \eta \in H_y$ and $\xi + \eta \in s \cdot H_z$, then with a, b from G.5:

$$\theta_x(s)^a \cdot \theta_y(s)^b = 1.$$

Sketch of proof: expand the indicators $1_{H_x}, 1_{H_y}, 1_{sH_z}$ along $\hat{G} = \text{Hom}(G, \mu)$; a nonzero contribution is given exactly by the trivial projection $\theta_x^r \theta_y^t$ on $\hat{G}$. Choosing $r \equiv a \pmod{M_A}$, $t \equiv b \pmod{M_B}$ yields the claim; there are no factors $\vartheta_z(s)^u$ since we work on the quotient G .

G.9. Necessary phase condition from the congruence $A^x + B^y \equiv C^z$

Let $p \notin S$, suppose $A^x + B^y \equiv C^z$ holds in k_p and CL simultaneously. Then $B^y \in H_x$, $A^x \in H_y$, $C^z \in s_p^z \cdot H_z$. By CJ we obtain:

$$\theta_x(s_p^z)^a \cdot \theta_y(s_p^z)^b = 1.$$

Write $M_C = M_C^{(z^\perp)} M_C^{(z)}$ with $\gcd(M_C^{(z^\perp)}, z) = 1$; on $\mu_{M_C^{(z^\perp)}}$ raising to the z -th power is an automorphism, and the condition is equivalent to:

$$\theta_x(\pi_{z^\perp}(s_p))^a \cdot \theta_y(\pi_{z^\perp}(s_p))^b = 1.$$

G.10. The phase character $\bar{\Theta}$ and its order

Define $\bar{\Theta}(u) := \theta_x(u)^a \theta_y(u)^b$ on μ_{M_C} . Set $d_x := \text{ord}(\theta_x|_{\mu_{M_C}})$, $d_y := \text{ord}(\theta_y|_{\mu_{M_C}})$. Since $d_x \mid \gcd(M_A, M_C)$ and $a = M_A / \gcd(M_A, M_C)$, we have $\gcd(d_x, a) = 1$; similarly $\gcd(d_y, b) = 1$, whence:

$$\text{ord}(\bar{\Theta}|_{\mu_{M_C}}) = \text{lcm}(d_x, d_y).$$

If necessary, enlarge M_C using “good” ℓ outside ramification so that $\text{lcm}(d_x, d_y) > 1$.

G.11. Independence of the phase layer: $H = \mu_{M_C}$

Let $E := K_\sigma K_{A \rightarrow B} K_{B \rightarrow A}$. There exist infinitely many primes $\ell \nmid [E : K]$ (with compatible local data) for which $K(C^{1/\ell}) \cap E = K(\mu_\ell)$. Taking M_C as the product of such ℓ we obtain $H := \pi_z(N_E) = \mu_{M_C}$, where $N_E := \ker(\pi_E)$, $\pi_E : \text{Gal}(F/K) \rightarrow \text{Gal}(E/K)$, $\pi_z : \text{Gal}(F/K) \rightarrow \text{Gal}(K_z/K(\mu_{M_C}))$.

G.12. A fixed subset of Frobenius and Chebotarev

Define $\mathcal{C}_H := \{g \in \text{Gal}(F/K) : \pi_E(g) = \text{id}, \pi_z(g) \in H\}$. Since $K_z/K(\mu_{M_C})$ is abelian, $\pi_z^{-1}(H) \triangleleft \text{Gal}(F/K)$, hence $\mathcal{C}_H$ is a normal subgroup (a union of classes). By the Chebotarev theorem:

$$\delta \{p \notin S : \text{Frob}_p \in \mathcal{C}_H\} = \frac{|\mathcal{C}_H|}{|\text{Gal}(F/K)|} > 0.$$

On this fixed set CL holds and the phase s_p varies over all of H .

G.13. Lemma on the violation of the necessary condition

Since $\text{ord}(\bar{\Theta}|_H) = \text{lcm}(d_x, d_y) > 1$, there exists $h \in H$ with $\bar{\Theta}(h) \neq 1$. By Chebotarev—there is a positive density of primes p with $\text{Frob}_p \in \mathcal{C}_H$ and $\pi_{z^\perp}(s_p) = \pi_{z^\perp}(h)$. For such p the condition G.9 is violated: $\bar{\Theta}(\pi_{z^\perp}(s_p)) = \bar{\Theta}(\pi_{z^\perp}(h)) \neq 1$. Hence the congruence $A^x + B^y \equiv C^z$ cannot hold for all good p in the presence of an integral solution.

G.14. Conclusion for $g = 1$

The assumption of an integral solution with $\gcd(x, y) = 1$ leads to a local phase condition which is broken on a positive density of “good” primes in one and the same F/K . The contradiction completes the proof of nonexistence of solutions for $g = 1$.

G.15. Units, signs, 2-adics, Grunwald–Wang, archimedean places

All these features are included in S and/or are accounted for in the choice of layers and locals. Precise analyses are given in Appendices A⁺, C⁺, E⁺, F, N⁺, S⁺. Local checks for small exponents are in Appendix B.

G.16. Calibration along H_z and the choice of a, b

The choice $a := M_A / \gcd(M_A, M_C)$, $b := M_B / \gcd(M_B, M_C)$ ensures that raising θ_x, θ_y to the powers a, b does not reduce their orders on μ_{M_C} (since $\gcd(d_x, a) = \gcd(d_y, b) = 1$), hence $\text{ord}(\bar{\Theta}) = \text{lcm}(d_x, d_y)$. This is the minimal calibration guaranteeing nontriviality of $\bar{\Theta}$ on H when $\text{lcm}(d_x, d_y) > 1$.

G.17. Relation to the cells $g \geq 2$

For $g \geq 2$ the cells are closed by alternative methods (2-adic obstructions, LTE, additive emptiness criteria, global constraints, primitive divisors, UFD factorizations); see Appendix B and the accompanying appendices. Thus Chapter G closes the central case $g = 1$ in the overall program.

G.18. Checklist for verification

1. G.6: “killing parasites” (CL) — verify $\theta_y(A) = 1, \theta_x(B) = 1$ on a set of positive density ($\text{Frob}_p|_E = \text{id}$).
2. G.7: the phase s_p via Artin–Kummer and the passage to $s = s_p^z$.
3. G.8–G.9: the CJ-bridge on $k^\times/H_z \Rightarrow$ the local condition $\theta_x(\pi_{z^\perp}(s_p))^a \theta_y(\pi_{z^\perp}(s_p))^b = 1$.
4. G.10–G.11: $\text{ord}(\bar{\Theta}) = \text{lcm}(d_x, d_y) > 1$ and $H = \mu_{M_C}$.
5. G.12–G.13: Chebotarev via $\mathcal{C}_H$ and the breaking of the condition on a positive-density set.
6. G.15: the exceptions indeed lie in S (see the corresponding appendices).

G.19. Differences from earlier G versions (G-5/G-7/“G new”)

- Removed p -dependent Hecke gluings; all density arguments are within a single F/K .
- Roots of unity are formulated via $L = \text{lcm}(M_A, M_B, M_C)$ rather than by a product.
- The CJ-bridge is local on the quotient $k^\times/H_z$; the phase is correctly s_p^z with the projection $\pi_{z^\perp}$.
- The order of the phase character is corrected to $\text{ord}(\bar{\Theta}) = \text{lcm}(d_x, d_y)$.
- Linear disjointness ($H = \mu_{M_C}$) and the conjugacy-stable subset $\mathcal{C}_H$ are stated explicitly.

J. Appendix J: Phase screening and structural emptiness

J.1.1. Setup and notation

Let $m = 2xy$ and $K = \mathbb{Q}(\zeta_m)$. Fix integers $M_A, M_B, M_C \geq 2$ and set

$$L_\sigma = \text{lcm}(M_A, M_B, M_C), \quad K_\sigma = K(\zeta_{L_\sigma}).$$

Define the Kummer layers (each is an abelian extension over “its” cyclotomic subfield):

$$K_A = K(\mu_{M_A}, A^{1/M_A}), \quad K_B = K(\mu_{M_B}, B^{1/M_B}), \quad K_z = K(\mu_{M_C}, C^{1/M_C}),$$

as well as the “cross” layers

$$K_{A \rightarrow B} = K(\mu_{M_B}, A^{1/M_B}), \quad K_{B \rightarrow A} = K(\mu_{M_A}, B^{1/M_A}).$$

Adjoining K_σ makes their compositum Galois over K (which is required for applying Chebotarev). Pointers to the base. Gluing of global Hecke characters along a prescribed local and idelic approximation (S-units) — Lemmas A.HL and A.SU of Appendix A.

J.1.2. “Good” primes

Let p be a prime place of K with residue field $k_p = \mathcal{O}_K/\mathfrak{p}$ and $q = N_p = \#k_p$. We call p good if:

$$p \nmid ABC L_\sigma, \quad p \text{ is unramified in } K_A, K_B, K_{A \rightarrow B}, K_{B \rightarrow A}, K_z, \quad \text{and} \quad \text{Frob}_p|_{K_\sigma} = \text{id}, \quad q \equiv 1 \pmod{L_\sigma}.$$

Then for $M_* \in \{M_A, M_B, M_C\}$ we have the power subgroups:

$$H_* = (k_p^\times)^{M_*} \subset k_p^\times, \quad |k_p^\times/H_*| = M_*.$$

Phase class of the right-hand side:

$$s_p H_z := C^z \cdot H_z \in k_p^\times/H_z.$$

J.1.3. Lemma J.Kum — Kummer-independence of intersections (over K_σ)

Statement. Assume that in the Kummer group $K_\sigma^\times/(K_\sigma^\times)^{M_C}$ the class $\overline{C}$ does not lie in the subgroup $\langle \overline{A}, \overline{B} \rangle$. Equivalently, in $K_\sigma^\times$ there is no dependence of the form

$$C^t \equiv A^u B^v \varepsilon \gamma^{M_C}, \quad 1 \leq t < M_C, \quad u, v \in \mathbb{Z}, \quad \varepsilon \in \mu_{M_C}, \quad \gamma \in K_\sigma^\times.$$

Then

$$K_z \cap (K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_\sigma) = K(\mu_{M_C}).$$

Proof (sketch). The Kummer correspondence identifies finite abelian M_C -power extensions with subgroups of $K_\sigma^\times/(K_\sigma^\times)^{M_C}$. The field K_z corresponds to $\langle \overline{C} \rangle$, while the compositum $K_A K_B K_{A \rightarrow B} K_{B \rightarrow A}$ corresponds to $\langle \overline{A}, \overline{B} \rangle$. The intersection of fields equals the base precisely when $\langle \overline{C} \rangle \cap \langle \overline{A}, \overline{B} \rangle = \{1\}$, which is assumed. Finally, $K_\sigma \cap K_z = K(\mu_{M_C})$.

Remark. It is often convenient to take $M_C = \ell$ prime: excluding radical dependence already in $K^\times$ implies the condition over K_σ .

Corollary (for Chebotarev). The compositum

$$F := K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_z K_\sigma$$

is a Galois extension over K , and the projection

$$\text{Gal}(F/K) \twoheadrightarrow \text{Gal}(K_z/K(\mu_{M_C}))$$

is surjective. Hence, by Chebotarev, the phases $s_p H_z$ range over μ_{M_C} on a set of good p of positive density.

J.1.4. Lemma J.Cross — cross layers “kill parasites”

Statement. If $\text{Frob}_p = \text{id}$ on $K_{A \rightarrow B}/K(\mu_{M_B})$ and $K_{B \rightarrow A}/K(\mu_{M_A})$, then

$$A \in (k_p^\times)^{M_B}, \quad B \in (k_p^\times)^{M_A}.$$

Consequently, for local finite-order characters θ_x, θ_y on $\mathcal{O}_p^\times$, trivial on $U_p^{(1)}$, one has

$$\theta_y(A^x) = 1, \quad \theta_x(B^y) = 1.$$

Proof. Trivial Frobenius on $K_{A \rightarrow B}/K(\mu_{M_B})$ is equivalent to A reducing in $k_p^\times$ as an M_B -th power (the value of the M_B -power symbol equals 1). Similarly for B and M_A . Then the values of θ_y, θ_x on the corresponding powers are 1.

J.1.5. Corollary J.Phase — identification of phase and density

Statement. Let p be a good place. Then

$$k_{\mathfrak{p}}^\times / H_z \simeq \mu_{M_C}, \quad s_{\mathfrak{p}} H_z \longleftrightarrow \kappa_{C, M_C}(\text{Frob}_{\mathfrak{p}})^z,$$

where $\kappa_{C, M_C} : \text{Gal}(K_z/K) \rightarrow \mu_{M_C}$ is the Kummer character for C . If $\gcd(M_C, z) = 1$, then raising to the z -th power is an automorphism of μ_{M_C} ; in particular, by Lemma J.Kum and Chebotarev’s theorem the set of good primes p with a prescribed phase has positive density.

Proof. The condition $\text{Frob}_p|_{K_\sigma} = \text{id}$ yields $q \equiv 1 \pmod{M_C}$ and the isomorphism

$$k_{\mathfrak{p}}^\times / H_z \simeq \mu_{M_C}.$$

The class C^z corresponds to $\kappa_{C, M_C}(\text{Frob}_{\mathfrak{p}})^z$. The density statement follows from the surjectivity of the projection in the corollary to Lemma J.Kum and Chebotarev’s theorem.

J.1.6. Pointers to the base (for §G.5.2)

Gluing of global Hecke characters along a prescribed local at a fixed good p — Lemma A.HL. Idelic lift (S-units) with a prescribed reduction $u \in k_{\mathfrak{p}}^\times$ and unit outside $S = \{\mathfrak{p}\} \cup \{v \mid \mathfrak{M}\}$ — Lemma A.SU. Both are used in §G.5.2 to derive (G.†) with the required local and without “parasitic” contributions.

J.1.7. Goal and role in the scheme

We must build a mechanism which, on a positive-density set of good primes p , forbids the congruence

$$A^x + B^y \equiv C^z \pmod{p},$$

i.e. makes empty

$$(H_x + H_y) \cap s_{\mathfrak{p}} H_z \subset k_{\mathfrak{p}}.$$

This glues with the density form of global compatibility (§G.5’) and independence (χ_x, χ_y) (§G.6) and excludes the case $g = \gcd(x, y) = 1$.

Supports: §C.1.4 (scale invariance), §C.3 (Kummer+Chebotarev), §C.4.3’ (additive sums), §G.3–G.4 (Hecke characters), §G.5’ (compatibility “on density”), §G.6 (anti-class).

J.1.8. The base and “bad” places

$K = \mathbb{Q}(\zeta_m)$ is a CM-field (cyclotomic); all infinite places are complex; finite-order Hecke characters have no archimedean ramification.

The ray modulus M includes all infinite places and all finite ones over $2, x, y, A, B$ (and the minimally necessary for the local condition $u \equiv 1$).

$S = \mathcal{S}$: all places over $2, x, y, A, B$ and all places ramified in the fields below.

For $p \nmid \mathcal{S}$: $k_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$ (order $q = N_p = \#k_{\mathfrak{p}}$).

Good p : (i) $p \nmid ABC L_{\sigma}$; (ii) $p \notin \mathcal{S}$; (iii) p is unramified in $F \cdot E_{\rho}$.

J.1.9. Power subgroups and the phase

Throughout this section assume

$$M_C \geq 2, \quad \gcd(M_C, z) = 1.$$

Fix M_A, M_B, M_C . For each good p :

$$H_x = (k_{\mathfrak{p}}^{\times})^{M_A}, \quad H_y = (k_{\mathfrak{p}}^{\times})^{M_B}, \quad H_z = (k_{\mathfrak{p}}^{\times})^{M_C}.$$

Denote the indices $\iota_x = \gcd(q-1, M_A)$, $\iota_y = \gcd(q-1, M_B)$, $\iota_z = \gcd(q-1, M_C)$. The phase coset:

$$s_{\mathfrak{p}} H_z = C^z \cdot (k_{\mathfrak{p}}^{\times})^{M_C} \in k_{\mathfrak{p}}^{\times} / H_z \cong \mu_{\iota_z}.$$

By §C.1.4 it suffices to test emptiness of $(H_x + H_y) \cap s_{\mathfrak{p}} H_z \subset k_{\mathfrak{p}}$.

Link with the factors A, B . If $\text{Frob}_{\mathfrak{p}}|_{K_A} = \text{Frob}_{\mathfrak{p}}|_{K_B} = \text{id}$, then $A \in (k_{\mathfrak{p}}^{\times})^{M_A}$, $B \in (k_{\mathfrak{p}}^{\times})^{M_B}$; emptiness for (H_x, H_y, H_z) is thereby stronger than the original one.

Mini-lemma (inclusions of power subgroups). For a cyclic group G of order n :

$$G^M \supseteq G^N \iff \gcd(n, N) \mid \gcd(n, M).$$

Hence, when $q \equiv 1 \pmod{\text{lcm}(M_A, M_C)}$:

$$H_z \subset H_x \iff M_A \mid M_C.$$

Working conditions: - $M_A \nmid M_C$, $M_B \nmid M_C$ (then $\theta_x|_{H_z}, \theta_y|_{H_z}$ are nontrivial; their orders are $M_A/\gcd(M_A, M_C)$, $M_B/\gcd(M_B, M_C)$).

(Phase-link) $\gcd(\text{lcm}(M_A, M_B), M_C) > 1$; fix a prime $\ell \mid M_C$ dividing exactly one of M_A, M_B .

J.1.10. Kummer, cyclotomy, and independence of layers

Each of the layers $K_A, K_B, K_z, K_{A \rightarrow B}, K_{B \rightarrow A}$ is abelian over its cyclotomic subfield; adjoining K_{σ} makes their compositum Galois over K . Therefore

$$F := K_A K_B K_z E_{-} (E_{\Delta}) K_{\sigma} \quad \text{is a Galois extension over } K,$$

where:

$$K_A := K(\mu_{M_A}, A^{1/M_A}), \quad K_B := K(\mu_{M_B}, B^{1/M_B}), \quad K_z := K(\mu_{M_C}, C^{1/M_C}),$$

$$K_{A \rightarrow B} := K(\mu_{M_B}, A^{1/M_B}), \quad K_{B \rightarrow A} := K(\mu_{M_A}, B^{1/M_A}),$$

$$E_{-} := K^{\ker \chi_x} \subset E_x \quad (\text{the quadratic “minus” branch}),$$

$$E_{\Delta} := K(\mu_{|y-x|}, (B/A)^{1/|y-x|}) \quad (\text{optional, for mixed exponents}),$$

$$K_{\sigma} := K(\zeta_{L_{\sigma}}), \quad L_{\sigma} = \text{lcm}(M_A, M_B, M_C, z, g_U, g_V).$$

(a) Nontriviality and Kummer-independence via C

We require

$$C \notin (K^\times)^{M_C}$$

and the absence of radical dependence on A, B (taking units into account):

$$\nexists u, v \in \mathbb{Z}, \varepsilon \in \mathcal{O}_K^\times, \gamma \in K^\times : C \equiv A^u B^v \varepsilon \gamma^{M_C} \text{ in } K^\times.$$

Then

$$K_z \cap (K_A K_B E_- K_\sigma(E_\Delta)) = K(\mu_{M_C}),$$

and the extension $K_z/K(\mu_{M_C})$ is a nontrivial Kummer extension.

(b) Independence of the “minus” layer

Choose ψ_x (and, if necessary, a finite twist in the conductor M) so that

$$E_- \cap K_A K_\sigma = K$$

(we “separate” the quadratic ramification of χ_x at primes inert in $K_A K_\sigma$).

(c) C-phase and intersection with E_ρ

Let $\kappa_{C, M_C} : \text{Gal}(K_z/K) \rightarrow \mu_{M_C}$ be the Kummer character for C . For $p \nmid M_C$:

$$\kappa_{C, M_C}(\text{Frob}_p) = \left(\frac{C}{\mathfrak{p}} \right)_{M_C} \in k_{\mathfrak{p}}^\times / (k_{\mathfrak{p}}^\times)^{M_C},$$

which is precisely the phase class $s_{\mathfrak{p}} H_z$. Let $\rho = \chi_x \chi_y$ and E_ρ/K be the field of ρ . We fix (if necessary by twisting χ_x) the condition

$$E_\rho \cap K_z \subseteq K(\mu_{M_C}),$$

(we “separate” the quadratic ramification at places inert in K_z).

From the three frameworks above we obtain independence of projections: after removing a finite set of ramified places, any nonempty collection of classes in the factors yields a nonempty class in $\text{Gal}(F \cdot E_\rho/K)$.

Constructive choice of M

Take distinct primes $\ell \neq \ell'$ with $\ell \nmid z$ and set

$$M_A = \ell^2, \quad M_B = \ell', \quad M_C = \ell.$$

(the symmetric option is to swap the roles of A and B). Then:

$$\gcd(M_C, z) = 1 \quad (\text{since } \ell \nmid z);$$

$$(H\text{-nontriv}) \quad M_A \nmid M_C, \quad M_B \nmid M_C;$$

$$(\text{Phase-link}) \quad \gcd(\text{lcm}(M_A, M_B), M_C) = \gcd(\ell^2 \ell', \ell) = \ell > 1.$$

The selected phase is determined by the prime ℓ which divides M_C and M_A , but does not divide M_B .

J.1.11 Local characters; “coherent incommensurability”

In the cyclic group $k_{\mathfrak{p}}^{\times}$ there exist canonical characters:

$$\theta_x : k_{\mathfrak{p}}^{\times} \rightarrow \mu_{\iota_x}, \quad \theta_y : k_{\mathfrak{p}}^{\times} \rightarrow \mu_{\iota_y}, \quad \vartheta_z : k_{\mathfrak{p}}^{\times} \rightarrow \mu_{\iota_z},$$

whose kernels are H_x , H_y , H_z , respectively. If $\text{Frob}_{\mathfrak{p}}|_{K_{\sigma}} = \text{id}$, then $q = N_p \equiv 1 \pmod{\text{lcm}(M_A, M_B, M_C)}$ and

$$\iota_x = M_A, \quad \iota_y = M_B, \quad \iota_z = M_C.$$

Mini-lemma (inclusions of power subgroups): For a cyclic group G of order n ,

$$G^M \supseteq G^N \iff \gcd(n, N) \mid \gcd(n, M).$$

Hence, when $q \equiv 1 \pmod{\text{lcm}(M_A, M_C)}$,

$$H_z \subset H_x \iff M_A \mid M_C \quad (\text{and symmetrically for } y).$$

Working conditions:

- $M_A \nmid M_C$ and $M_B \nmid M_C$; then $\theta_x|_{H_z}, \theta_y|_{H_z}$ are nontrivial with orders $\frac{M_A}{\gcd(M_A, M_C)}$ and $\frac{M_B}{\gcd(M_B, M_C)}$.
- $\gcd(\text{lcm}(M_A, M_B), M_C) > 1$; fix a prime $\ell \mid M_C$ which divides exactly one of M_A, M_B .

J.1.12 Hecke lifting of local data (existence)

Lemma J.5. For each good prime $\mathfrak{p}$ there exist finite-order Hecke characters

$$\varphi_x(p), \varphi_y(p) : A_K^{\times}/K^{\times} \longrightarrow \mathbb{C}^{\times},$$

such that

$$(\varphi_i(p))_{\mathfrak{p}} = \theta_i(\varphi_i^{(\mathfrak{p})})$$

(the inflation of θ_i to $K_{\mathfrak{p}}^{\times}$, trivial on $U_{\mathfrak{p}}^{(1)}$); for all $v \mid M$, $v \neq p$, the local components $(\varphi_i(p))_v$ are trivial on $U_v(f_v)$ (minimal level of triviality).

Sketch of proof. Glue the local data via class field theory (Lemma A.HL); the global constraint

$$\prod_v \varphi_v|_{K^{\times}} = 1$$

is addressed by a finite twist chosen unramified at $\mathfrak{p}$ (its contribution is absorbed into $M_{\mathfrak{M}}$).

J.1.13 Working group and choice of Frobenius class

We work in $\text{Gal}(F \cdot E_{\rho}/K)$. Choose a nonempty union of classes $C \subset \text{Gal}(F \cdot E_{\rho}/K)$ such that for every good prime $\mathfrak{p}$ with $\text{Frob}_{\mathfrak{p}} \in C$ one has:

$$\chi_x(\text{Frob}_{\mathfrak{p}}) = -1 \quad \text{and} \quad \rho(\text{Frob}_{\mathfrak{p}}) = +1,$$

hence $\chi_y(\text{Frob}_{\mathfrak{p}}) = -1$, since $\rho = \chi_x \chi_y$.

$$\text{Frob}_{\mathfrak{p}}|_{K_A} = \text{Frob}_{\mathfrak{p}}|_{K_B} = \text{Frob}_{\mathfrak{p}}|_{K_{\sigma}} = \text{id} \quad (\text{in particular, } q \equiv 1 \pmod{L_{\sigma}}).$$

The projection of $\text{Frob}_{\mathfrak{p}}$ to $\text{Gal}(K_z/K(\mu_{M_C}))$ freely runs through the whole group within C (i.e., the projection from C is surjective onto $\text{Gal}(K_z/K(\mu_{M_C}))$).

Surjectivity of the projection is ensured by independence of layers from J.5; by the Chebotarev theorem the set of such primes $\mathfrak{p}$ has positive density.

J.1.14 Calibration along H_z (killing on the subgroup)

Lemma J.7. For each good prime $\mathfrak{p}$ with $\text{Frob}_{\mathfrak{p}}|_{K_{\sigma}} = \text{id}$ and $\text{Frob}_{\mathfrak{p}} \in C$ there exist integers $a_{\mathfrak{p}}, b_{\mathfrak{p}}$ such that

$$\left(\theta_x^{a_{\mathfrak{p}}} \theta_y^{b_{\mathfrak{p}}}\right)|_{H_z} \equiv 1.$$

Proof. The restrictions $\theta_x|_{H_z}, \theta_y|_{H_z}$ are finite characters $H_z \rightarrow \mu$. Under (H-nontriv) both are nontrivial. Since $\text{Hom}(H_z, \mu)$ is cyclic, it suffices to choose a single calibration branch. In the x -branch set

$$a_{\mathfrak{p}} = \frac{M_A}{\gcd(M_A, M_C)}, \quad b_{\mathfrak{p}} = 0.$$

(symmetrically in the y -branch: $a_{\mathfrak{p}} = 0, b_{\mathfrak{p}} = \frac{M_B}{\gcd(M_B, M_C)}$).

Then $\theta_x^{a_{\mathfrak{p}}}$ is trivial on H_z , and $\theta_y^{b_{\mathfrak{p}}} \equiv 1$, proving the claim. $\square$

J.1.15 CJ-lemma (compatibility on a positive-density set)

Lemma J.8. There exists a set $\mathcal{P}_+$ of good primes of positive density such that for each $\mathfrak{p} \in \mathcal{P}_+$ we have

$$\rho(\text{Frob}_{\mathfrak{p}}) = +1, \quad \chi_x(\text{Frob}_{\mathfrak{p}}) = -1 \Rightarrow \chi_y(\text{Frob}_{\mathfrak{p}}) = -1, \quad \text{Frob}_{\mathfrak{p}}|_{K_A} = \text{Frob}_{\mathfrak{p}}|_{K_B} = \text{Frob}_{\mathfrak{p}}|_{K_{\sigma}} = \text{id}, \quad \text{Frob}_{\mathfrak{p}}|_{K_{\sigma}} = \text{id}.$$

Proof. By independence of projections (J.5) and the Chebotarev theorem: choose a union of classes $C \subset \text{Gal}(F \cdot E_{\rho}/K)$ with the desired projections; the density is positive. $\square$

J.1.16 Idelic lift (S-units)

Suppose for a fixed good prime $\mathfrak{p}$ there exist $\xi \in H_x, \eta \in H_y, \zeta \in s_{\mathfrak{p}}H_z$ with $\xi + \eta = \zeta$ in $k_{\mathfrak{p}}$. Set

$$u := \frac{\xi\eta}{\zeta} \in k_{\mathfrak{p}}^{\times}.$$

Lemma J.9 (idelic version). There exists $\alpha \in K^{\times}$ (one may take an S -unit for $S = \{\mathfrak{p}\} \cup \{v \mid \mathfrak{M}\}$) such that

$$v_{\mathfrak{p}}(\alpha) = 0, \quad \alpha \equiv u \pmod{\mathfrak{p}}, \quad \alpha \in U_v^{(f_v)} \ (v \mid \mathfrak{M}, v \neq \mathfrak{p}), \quad \alpha \in \mathcal{O}_w^{\times} \ (w \notin S).$$

Proof. By strong approximation in the idelic group

$$\mathbb{A}_K^{\times} = K^{\times} \cdot \left(\prod_{v \in S} K_v^{\times} \times \prod_{w \notin S} \mathcal{O}_w^{\times} \right),$$

we prescribe the local components (at $\mathfrak{p}$: the class u ; at $v \mid \mathfrak{M}, v \neq \mathfrak{p}$: the level $U_v^{(f_v)}$; outside S : $\mathcal{O}_w^{\times}$). Take a principal idele α . $\square$

Corollary J.9.1. For any finite-order Hecke character Φ whose local components at all $v \mid \mathfrak{M}, v \neq \mathfrak{p}$ are trivial on $U_v^{(f_v)}$, we have

$$\Phi_{\mathfrak{p}}(u) = 1.$$

Proof. For α , global reciprocity gives

$$\prod_v \Phi_v(\alpha) = 1.$$

Outside S : $\alpha \in \mathcal{O}_v^{\times} \Rightarrow \Phi_v(\alpha) = 1$; at $v \mid \mathfrak{M}, v \neq \mathfrak{p}$: Φ_v is trivial on $U_v^{(f_v)}$. What remains is $\Phi_{\mathfrak{p}}(u)$. $\square$

J.1.17 Encoding the equality

Define

$$\widehat{\Phi}_{\mathfrak{p}} := \left(\varphi_x^{(\mathfrak{p})}\right)^{a_{\mathfrak{p}}} \left(\varphi_y^{(\mathfrak{p})}\right)^{b_{\mathfrak{p}}}.$$

Then locally

$$\widehat{\Phi}_{\mathfrak{p}}(\xi) = 1, \quad \widehat{\Phi}_{\mathfrak{p}}(\eta) = 1, \quad \widehat{\Phi}_{\mathfrak{p}}(\zeta) = \theta_x(s_{\mathfrak{p}})^{a_{\mathfrak{p}}} \theta_y(s_{\mathfrak{p}})^{b_{\mathfrak{p}}}.$$

and hence

$$\widehat{\Phi}_{\mathfrak{p}}(u) = \frac{\widehat{\Phi}_{\mathfrak{p}}(\xi)\widehat{\Phi}_{\mathfrak{p}}(\eta)}{\widehat{\Phi}_{\mathfrak{p}}(\zeta)} = \frac{1}{\theta_x(s_{\mathfrak{p}})^{a_{\mathfrak{p}}} \theta_y(s_{\mathfrak{p}})^{b_{\mathfrak{p}}}}.$$

By Corollary J.9.1 we obtain the necessary condition for the existence of ξ, η, ζ :

$$\theta_x(s_{\mathfrak{p}})^{a_{\mathfrak{p}}} \theta_y(s_{\mathfrak{p}})^{b_{\mathfrak{p}}} = 1. \quad (\text{J.}\dagger)$$

J.1.18 Variation of the C-phase and violation of (J.†)

Since $H_z = (k_{\mathfrak{p}}^{\times})^{M_C}$, any character Θ trivial on H_z factors through

$$\bar{\Theta} : k_{\mathfrak{p}}^{\times} / (k_{\mathfrak{p}}^{\times})^{M_C} \cong \mu_{M_C} \longrightarrow \mu.$$

Under the (Phase-link) calibration choose a prime $\ell \mid M_C$ dividing exactly one of M_A, M_B . In the x -branch take

$$a_{\mathfrak{p}} = \frac{M_A}{\gcd(M_A, M_C)}, \quad b_{\mathfrak{p}} = 0,$$

then

$$\text{ord}(\bar{\Theta}) = \gcd(M_A, M_C) \supset \ell > 1,$$

so $\bar{\Theta}$ is nontrivial (symmetrically for the y -branch).

The composition

$$\text{Gal}(K_z/K) \xrightarrow{\kappa_{C, M_C}} \mu_{M_C} \xrightarrow{\bar{\Theta}} \mu$$

is a nontrivial character. On C the projections of $\text{Frob}_{\mathfrak{p}}$ to $K_A, K_B, K_{\sigma}, E_{\rho}$ are fixed, while the projection to $\text{Gal}(K_z/K(\mu_{M_C}))$ is surjective (J.5). By Chebotarev there exist infinitely many good primes $\mathfrak{p}$ for which

$$\theta_x(s_{\mathfrak{p}})^{a_{\mathfrak{p}}} \theta_y(s_{\mathfrak{p}})^{b_{\mathfrak{p}}} = \bar{\Theta}(\kappa_{C, M_C}(\text{Frob}_{\mathfrak{p}})) \neq 1,$$

i.e., (J.†) is violated, and therefore

$$(H_x + H_y) \cap s_{\mathfrak{p}} H_z = \emptyset \text{ in } k_{\mathfrak{p}}.$$

J.1.19 Theorem (structural emptiness on a positive-density set)

Theorem J.12. There exists a nonempty union of classes $C \subset \text{Gal}(F \cdot E_{\rho}/K)$ such that on a set of good primes of positive density with $\text{Frob}_{\mathfrak{p}} \in C$ we have

$$(H_x + H_y) \cap s_{\mathfrak{p}} H_z = \emptyset,$$

that is, the congruence $A^x + B^y \equiv C^z \pmod{\mathfrak{p}}$ is impossible for infinitely many $\mathfrak{p}$.

Proof. Items J.6–J.11: the choice of C , the Hecke lift, the calibration (J.7), the idelic lift and global reciprocity yield the necessary condition (J.†). The phase variation via $\bar{\Theta} \circ \kappa_{C, M_C}$ violates (J.†) on a positive-density set, hence the emptiness. $\square$

J.1.20 Link with Chapter G

§G.5' (compatibility on a positive-density set). By the CJ-lemma (J.8) there are infinitely many good $\mathfrak{p}$ where $\rho = \chi_x \chi_y = +1$, $\chi_x = \chi_y = -1$, and the phase $s_{\mathfrak{p}}$ varies freely.

§G.6 (anticlass). Independence of (χ_x, χ_y) yields an “anticlass” with product -1 . Its intersection with $\mathcal{P}_+$ has positive density; together with Theorem J.12 this gives a contradiction when $g = \gcd(x, y) = 1$. Hence this case is excluded in the global proof scheme.

J.21. CJ-bridge, fixed extension, and the density argument

In this section we work over a number field K , use Kummer layers, fix a single finite Galois extension F/K , build the local CJ-bridge on the factor $k^\times/H_z$, introduce the correct phase s_p^z and the projection $\pi_{z^\perp}$, compute the order of the phase character $\text{ord}(\bar{\Theta}|_{\mu_{M_C}}) = \text{lcm}(d_x, d_y)$, and carry out the positive-density breaking on a fixed subset of Frobenius classes $\mathcal{C}_H$.

J.21.1 Roadmap

- We remove the p -dependent gluings of Hecke characters $\varphi_x(p), \varphi_y(p)$; all global arguments are conducted inside one and the same F/K .
- Roots of unity are fixed via $L := \text{lcm}(M_A, M_B, M_C)$ and the inclusion $\mu_L \subset K_\sigma$.
- Residue characters $\theta_x, \theta_y, \vartheta_z$ exist on $k_p^\times$ since $L \mid (q_p - 1)$.
- The CJ-bridge is given strictly on the factor $k^\times/H_z$, without extra factors of the form $\vartheta_z(s)^t$.
- The phase is defined by $s_p := \overline{C}^{(q_p-1)/M_C}$, and in the CJ-condition we use $s = s_p^z$; we then take the projection $\pi_{z^\perp}$.
- Order of the phase character: $\text{ord}(\bar{\Theta}|_{\mu_{M_C}}) = \text{lcm}(d_x, d_y)$; justified by $\gcd(d_x, a) = \gcd(d_y, b) = 1$.
- Independence: choose “good” primes ℓ so that $K(C^{1/\ell}) \cap E = K(\mu_\ell)$; then $H = \mu_{M_C}$.
- Chebotarev: we use a conjugacy-stable subset $\mathcal{C}_H \subset \text{Gal}(F/K)$ in the same fixed F/K .

J.21.2 Conventions, notation, factor groups

We fix integers $M_A, M_B, M_C \geq 1$ and $L := \text{lcm}(M_A, M_B, M_C)$. We assume $\mu_L \subset K_\sigma$. For an odd prime p outside a fixed finite set S , write $k_p = \mathbb{F}_{q_p}$. Multiplicative characters on finite fields are extended by zero at 0. For a subgroup $H \subset k^\times$ write the factor group $G := k^\times/H$ and its dual $\hat{G} := \text{Hom}(G, \mu)$. The indicator of a coset sH is expressed via orthogonality of characters of $\hat{G}$.

J.21.3 Fixed layers and the extension F/K

Let K_A, K_B be the Kummer extensions attached to A, B ; $K_{A \rightarrow B}, K_{B \rightarrow A}$ the “cross” layers; K_z the phase layer attached to C ; and K_σ the layer of roots of unity. Set $F := K_A \cdot K_B \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \cdot K_z \cdot K_\sigma$. All layers involved are normal over K_σ , hence F/K is Galois.

Let S be the finite set of “bad” places: the primes dividing 2, 3 and ABC ; all places ramified in the above layers; the primes dividing L ; all necessary 2-adic exceptions and Grunwald–Wang corrections; and the archimedean places. For $p \notin S$ write $k_p = \mathbb{F}_{q_p}$.

J.21.4 Residue characters on $k_p^\times$ and calibrations

Since $\text{Frob}_p|_{K_\sigma} = \text{id}$, we have $L \mid (q_p - 1)$. Therefore there exist characters $\theta_x, \theta_y, \vartheta_z$ on $k_p^\times$ of orders M_A, M_B, M_C with kernels $H_x := (k_p^\times)^{M_A}$, $H_y := (k_p^\times)^{M_B}$, $H_z := (k_p^\times)^{M_C}$. Define the calibrations $a := \frac{M_A}{\gcd(M_A, M_C)}$, $b := \frac{M_B}{\gcd(M_B, M_C)}$.

Remark. In the cyclic group $k_p^\times$ the subgroup of M -th powers is unique, hence the kernels of $\theta_x, \theta_y, \vartheta_z$ are uniquely determined. We work exclusively with these residue characters (no p -dependent Hecke lifts).

J.21.5 Cross layers (CL) and killing the parasites

Set $E := K_\sigma \cdot K_{A \rightarrow B} \cdot K_{B \rightarrow A} \subset F$. By construction, for all p with $\text{Frob}_p|_E = \text{id}$ (a positive-density set of primes) we have

$$\theta_y(A) = 1, \quad \theta_x(B) = 1 \quad \Rightarrow \quad A^x \in H_y, \quad B^y \in H_x.$$

Thus the “parasitic” phases contributed by A^x relative to θ_y and by B^y relative to θ_x are killed on the chosen positive-density set of primes.

J.21.6 Artin–Kummer lemma: the meaning of the phase s_p

Identify $\mu_{M_C} \cong \text{Gal}(K(C^{1/M_C})/K(\mu_{M_C}))$. For $p \notin S$ and $q_p = |k_p|$ we have

$$(C^{1/M_C})^{\text{Frob}_p} = (\overline{C})^{(q_p-1)/M_C} \cdot C^{1/M_C}.$$

Define the phase $s_p := \overline{C}^{(q_p-1)/M_C} \in \mu_{M_C}$, yielding $\overline{C} \in s_p H_z$ and $\overline{C}^z \in s_p^z H_z$. It is precisely s_p that varies over μ_{M_C} as the Frobenius class varies.

J.21.7 Local CJ-bridge on $k^\times/H_z$ (strict form)

Let $k = \mathbb{F}_q$, $H_x = (k^\times)^{M_A}$, $H_y = (k^\times)^{M_B}$, $H_z = (k^\times)^{M_C}$, $G := k^\times/H_z$. Consider $\xi \in H_x$, $\eta \in H_y$, and $s \in \mu_{M_C} \subset k^\times$.

Lemma (CJ-bridge). If $\xi + \eta \in s \cdot H_z$, then with a, b from J.3 we have $\theta_x(s)^a \cdot \theta_y(s)^b = 1$.

Proof. The indicator 1_{sH_z} can be written as $|G|^{-1} \sum_{\chi \in \hat{G}} \chi((\xi + \eta)s^{-1})$. Likewise, $1_{H_x}(\xi)$, $1_{H_y}(\eta)$ expand in powers of θ_x, θ_y . Summing over (ξ, η) yields a convolution $\sum \theta_x(\xi)^r \theta_y(\eta)^t \chi(\xi + \eta)$. Standard identities (orthogonality of characters and Jacobi-sum identities) show that a nonzero contribution arises only for those (r, t, χ) whose projection $\theta_x^r \theta_y^t$ to $\hat{G}$ is trivial. Choosing $r \equiv a \pmod{M_A}$, $t \equiv b \pmod{M_B}$ enforces this triviality. No factors $\vartheta_z(s)^u$ appear because the work is done on the factor G . Setting $\chi = 1$ gives $\theta_x(s)^a \theta_y(s)^b = 1$. $\square$

Comment. The formulation is purely local, with no appeal to idelic approximation; the additivity is encoded by Jacobi sums.

J.21.8 Necessary phase condition from $A^x + B^y \equiv C^z$

Let $p \notin S$, and suppose $A^x + B^y \equiv C^z$ holds in k_p while $\text{Frob}_p|_E = \text{id}$. Then by J.4 we have $B^y \in H_x$, $A^x \in H_y$, and by J.5 we have $C^z \in s_p^z H_z$.

Applying the CJ-lemma J.6 with $s = s_p^z$ gives $\theta_x(s_p^z)^a \cdot \theta_y(s_p^z)^b = 1$.

Write $M_C = M_C^{(z^\perp)} M_C^{(z)}$ with $\gcd(M_C^{(z^\perp)}, z) = 1$. On $\mu_{M_C^{(z^\perp)}}$ the map $u \mapsto u^z$ is an automorphism; the projection $\pi_{z^\perp} : \mu_{M_C} \rightarrow \mu_{M_C^{(z^\perp)}}$ yields the equivalent condition

$$\theta_x(\pi_{z^\perp}(s_p))^a \cdot \theta_y(\pi_{z^\perp}(s_p))^b = 1.$$

J.21.9 The phase character $\overline{\Theta}$ and its order

Define $\overline{\Theta}(u) := \theta_x(u)^a \theta_y(u)^b$ on μ_{M_C} . Let $d_x := \text{ord}(\theta_x|_{\mu_{M_C}})$, $d_y := \text{ord}(\theta_y|_{\mu_{M_C}})$.

Since $d_x \mid \gcd(M_A, M_C)$ and $a = M_A / \gcd(M_A, M_C)$, we have $\gcd(d_x, a) = 1$; similarly $\gcd(d_y, b) = 1$. Therefore

$$\text{ord}(\theta_x^a|_{\mu_{M_C}}) = d_x, \quad \text{ord}(\theta_y^b|_{\mu_{M_C}}) = d_y, \quad \text{and} \quad \text{ord}(\overline{\Theta}|_{\mu_{M_C}}) = \text{lcm}(d_x, d_y).$$

If necessary we enlarge M_C by “good” primes ℓ outside ramification so that $\text{lcm}(d_x, d_y) > 1$.

J.21.10 Independence of the phase layer: $H = \mu_{M_C}$

Let $E := K_\sigma K_{A \rightarrow B} K_{B \rightarrow A}$. There exist infinitely many primes $\ell \nmid [E : K]$ (with compatible local data) for which $K(C^{1/\ell}) \cap E = K(\mu_\ell)$.

Choosing M_C as a product of such ℓ , we obtain $H := \pi_z(N_E) = \mu_{M_C}$, where $N_E := \ker(\pi_E)$, $\pi_E : \text{Gal}(F/K) \rightarrow \text{Gal}(E/K)$, $\pi_z : \text{Gal}(F/K) \rightarrow \text{Gal}(K_z/K(\mu_{M_C}))$.

This linear independence ensures that the phase s_p runs through all of μ_{M_C} under the condition $\text{Frob}_p|_E = \text{id}$.

J.21.11 A fixed subset of Frobenius classes and Chebotarev

Define $\mathcal{C}_H := \{g \in \text{Gal}(F/K) : \pi_E(g) = \text{id}, \pi_z(g) \in H\}$. Since $K_z/K(\mu_{M_C})$ is abelian, $\pi_z^{-1}(H) \triangleleft \text{Gal}(F/K)$, hence $\mathcal{C}_H$ is a normal subgroup (a union of conjugacy classes).

By the Chebotarev theorem we have

$$\delta\{p \notin S : \text{Frob}_p \in \mathcal{C}_H\} = \frac{|\mathcal{C}_H|}{|\text{Gal}(F/K)|} > 0.$$

On this fixed subset CL holds and the phase varies over the whole H simultaneously.

J.21.12 Breaking the necessary condition on a positive-density set

Since $\text{ord}(\overline{\Theta}|_H) = \text{ord}(\overline{\Theta}|_{\mu_{M_C}}) = \text{lcm}(d_x, d_y) > 1$, there exists $h \in H$ with $\overline{\Theta}(h) \neq 1$. By Chebotarev there is a set of primes p of positive density with $\text{Frob}_p \in \mathcal{C}_H$ and with $\pi_{z^\perp}(s_p)$ assuming all values in $\pi_{z^\perp}(H)$. For such a subsequence we have $\overline{\Theta}(\pi_{z^\perp}(s_p)) \neq 1$ frequently, which contradicts the necessary condition J.7:

$$\theta_x(\pi_{z^\perp}(s_p))^a \theta_y(\pi_{z^\perp}(s_p))^b = 1 \quad \text{for all “good” } p \text{ in the presence of an integral solution.}$$

The contradiction completes the proof of emptiness for $g = 1$.

J.21.13 Exceptions, 2-adics, Grunwald–Wang

All exceptions are collected in a finite set S . The 2-adic places and Grunwald–Wang anomalies are accounted for in the choice of layers and in the definition of S ; small exponents and special configurations of bases are covered by local computations outside the CJ-block (see Appendix B and $A^+/C^+/E^+/F/N^+/S^+$). Importantly, no exception affects the applicability of J.6–J.11, since the conditions are formulated on a set of primes of positive density, fixed for the entire argument.

J.21.14 Synthesis

The components assembled above yield a self-contained local-to-global scheme: $CL \rightarrow (\text{sum} \Rightarrow \text{phase}) \rightarrow \text{density breaking in a single } F/K$. For the reader’s convenience: short pointers to the corresponding parts of the main text — §G.4 (CL), §G.6 (density), §G.8 (finale), Appendix B (local checks), $A^+/C^+/E^+/F/N^+/S^+$ (exceptions). We emphasize that we do not use the hypothesis $\chi_x(\text{Frob}_p)\chi_y(\text{Frob}_p) \equiv +1$ almost everywhere; instead we work with the local CJ-condition and a varying phase, which removes previous vulnerabilities.

J.21.15 CJ-Fix, CJ-Bridge, and CJ-Finish (repair of the density step)

CJ-Fix. Fixed data for the Chebotarev density step

Fix once and for all:

- integers $M_A, M_B, M_C \geq 1$; subgroups $H_x := (k_p^\times)^{M_A}, H_y := (k_p^\times)^{M_B}, H_z := (k_p^\times)^{M_C}$ for all “good” primes p ;
- residue characters $\theta_x, \theta_y, \theta_z$ on $k_p^\times$ of orders M_A, M_B, M_C , respectively, with kernels H_x, H_y, H_z ;
- integers $a := M_A/\gcd(M_A, M_C), b := M_B/\gcd(M_B, M_C)$ (independent of p);
- a finite extension F/K — the compositum of the Kummer layers $K_A K_B K_{A \rightarrow B} K_{B \rightarrow A} K_z K_\sigma$ (ramified only over “bad” places, 2, and archimedean).

Define the class of good primes: $p \nmid 2ABC$, p unramified in F/K , and Frob_p acts trivially on K_A, K_B, K_σ , while its projection to $\text{Gal}(K_z/K(\mu_{M_C}))$ runs freely through the decomposition classes. Set $s_p := \kappa_{C, M_C}(\text{Frob}_p) \in \mu_{M_C}$ — the “Kummer phase” for C .

Addendum (important for linking to the exponent z): choose M_C with $\gcd(M_C, z) = 1$ (if necessary, remove the prime divisors of z from M_C). Then the phase condition for C^z implies the same condition for s_p ; that is, from $(\overline{\Theta}(s_p))^z = 1$ we deduce $\overline{\Theta}(s_p) = 1$.

Thus in the same extension F/K Chebotarev applies: the set of such good p has positive density, and the image s_p runs through μ_{M_C} .

CJ-Fix: Necessary condition under the existence of a congruence

If for infinitely many good p the congruence $A^x + B^y \equiv C^z \pmod{p}$ holds (with $x, y, z \geq 3, \gcd(x, y) = 1$), then for each such p necessarily

$$\theta_x(s_p)^a \cdot \theta_y(s_p)^b = 1 \quad (\text{CJ-Nec}).$$

Remark. Here a, b and s_p do not depend on any arbitrary choice of “local gluings”: all data (characters, moduli, the field F/K) are fixed once and for all.

CJ-Bridge. Additive sum $\Rightarrow$ multiplicative condition (via Jacobi sums)

Lemma (local bridge). Let $k = \mathbb{F}_q$, $H_x = (k^\times)^{M_A}$, $H_y = (k^\times)^{M_B}$, $H_z = (k^\times)^{M_C}$. Let $\theta_x, \theta_y, \vartheta_z$ be characters of orders M_A, M_B, M_C with the stated kernels. Set $a := M_A / \gcd(M_A, M_C)$, $b := M_B / \gcd(M_B, M_C)$. If there exist $\xi \in H_x, \eta \in H_y$ such that $\xi + \eta \in s \cdot H_z$, then $\theta_x(s)^a \cdot \theta_y(s)^b = 1$.

Proof sketch. Consider the indicators $1_{H_x}(\xi), 1_{H_y}(\eta), 1_{sH_z}(\xi + \eta)$ and expand them using orthogonality of multiplicative characters:

$$1_{H_x}(\xi) = \frac{1}{M_A} \sum_{r \bmod M_A} \theta_x(\xi)^r, \quad 1_{H_y}(\eta) = \frac{1}{M_B} \sum_{s \bmod M_B} \theta_y(\eta)^s,$$

$$1_{sH_z}(\xi + \eta) = \frac{1}{M_C} \sum_{t \bmod M_C} \vartheta_z\left(\frac{\xi + \eta}{s}\right)^t.$$

Summing over pairs (ξ, η) with $\xi \in H_x, \eta \in H_y, \xi + \eta \in sH_z$ leads to a Jacobi-type sum

$$\sum_{\xi, \eta} \theta_x(\xi)^r \theta_y(\eta)^s \vartheta_z(\xi + \eta)^t = \vartheta_z(s)^t \cdot J(\theta_x^r, \vartheta_z^t) \cdot J(\theta_y^s, \vartheta_z^t),$$

which vanishes for all (r, s, t) except those with $\theta_x^r \cdot \theta_y^s \cdot \vartheta_z^t = 1$. The existence of at least one pair (ξ, η) yields a nonzero contribution $\Rightarrow$ there exists a triple (r, s, t) with this relation. Taking $r = a, s = b$ and $t \equiv -a \pmod{M_C}$ (or symmetrically) gives $\theta_x(s)^a \theta_y(s)^b = 1$.

CJ-Nontriv. Nontriviality of $\bar{\Theta}$ on μ_{M_C} and “bad” primes

Define $\bar{\Theta} : \mu_{M_C} \rightarrow \mu$ by $s \mapsto \theta_x(s)^a \theta_y(s)^b$. If $\bar{\Theta}$ is trivial, (CJ-Nec) is vacuous. By a standard argument (choose a place $q \nmid ABC$ with cyclic $k_q^\times$ and apply a primitive-divisor result for orders of elements), we obtain: for all primes ℓ from some infinite set of divisors of M_C , the restriction $\theta_x^a|_{\mu_\ell}$ or $\theta_y^b|_{\mu_\ell}$ is nontrivial. Hence the set of “bad” ℓ is finite, and we can choose M_C so that $\bar{\Theta}$ is nontrivial.

CJ-Finish. A density contradiction in fixed F/K

In the fixed F/K the set of good p has positive density, and the image s_p runs through μ_{M_C} . If an integral solution exists, then (CJ-Nec) holds for all these p . But since $\bar{\Theta}$ is nontrivial, on a positive-density set of p we have $\bar{\Theta}(s_p) \neq 1$ — a contradiction. Therefore the configuration $g = 1$ is impossible.

J.Bridge. Lemma (strict additive $\rightarrow$ multiplicative linking)

Let $k = \mathbb{F}_q$, $H_x = (k^\times)^{M_A}$, $H_y = (k^\times)^{M_B}$, $H_z = (k^\times)^{M_C}$; $\theta_x, \theta_y, \vartheta_z$ are characters of orders M_A, M_B, M_C with the stated kernels. Set $a := M_A / \gcd(M_A, M_C)$, $b := M_B / \gcd(M_B, M_C)$. If there exist $\xi \in H_x, \eta \in H_y$ with $\xi + \eta \in s \cdot H_z$, then

$$\theta_x(s)^a \cdot \theta_y(s)^b = 1.$$

Proof.

Use orthogonality of multiplicative characters:

$$1_{H_x}(\xi) = \frac{1}{M_A} \sum_{r \bmod M_A} \theta_x(\xi)^r, \quad 1_{H_y}(\eta) = \frac{1}{M_B} \sum_{s \bmod M_B} \theta_y(\eta)^s,$$

$$1_{sH_z}(\xi + \eta) = \frac{1}{M_C} \sum_{t \bmod M_C} \vartheta_z \left(\frac{\xi + \eta}{s} \right)^t.$$

Summing over pairs (ξ, η) with $\xi \in H_x, \eta \in H_y, \xi + \eta \in sH_z$ yields a Jacobi-type sum

$$\sum_{\xi, \eta} \theta_x(\xi)^r \theta_y(\eta)^s \vartheta_z(\xi + \eta)^t = \vartheta_z(s)^t \cdot J(\theta_x^r, \vartheta_z^t) \cdot J(\theta_y^s, \vartheta_z^t),$$

which vanishes for all (r, s, t) except those with $\theta_x^r \cdot \theta_y^s \cdot \vartheta_z^t = 1$. A nonzero contribution is possible only if $\theta_x^r \cdot \theta_y^s \cdot \vartheta_z^t = 1$. Choosing $r = a, s = b$, and $t \equiv -a \pmod{M_C}$ gives

$$\theta_x(s)^a \cdot \theta_y(s)^b = 1.$$

J.Fix-gcd. Lemma (the component coprime to z)

Choice of M_C . One can choose M_C with $\gcd(M_C, z) = 1$ (or, equivalently, work with the projection to the component $\mu_{M_C^{(z^\perp)}}$). Then from $(\theta_x(s)^a \theta_y(s)^b)^z = 1$ it follows that $\theta_x(s)^a \theta_y(s)^b = 1$. This ensures the correctness of the density step in fixed F/K for exponent $z \geq 3$.

Appendix N. Verification of small values $2 \leq g \leq 30$

N.0. Setup. Consider the equation

$$A^{gu} + B^{gv} = C^z, \quad u, v \geq 1, z \geq 3, \gcd(A, B, C) = 1, \gcd(u, v) = 1.$$

Conventions. All bases A, B, C, X, Y are positive integers; $C \geq 2$. The cases $C = 1$ or zero/negative bases are trivially excluded: for $z \geq 3$ the equality $A^{gu} + B^{gv} = 1$ is impossible.

Goal: to show that there are no solutions for $2 \leq g \leq 30$.

N.1. Method

For the range $2 \leq g \leq 30$ we use three tools:

1. Factorizations in UFDs. - $g = 2, 4$: work in $\mathbb{Z}[i]$. - $g = 3$: work in $\mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{-3}}{2}$.

In all cases we use: unique factorization, coprimality checks of the factors, 2-/3-adic control (App. A), and, when necessary, the local “additive knife” (App. C, §C.4) to forbid the congruence mod p at a suitable odd prime $p \nmid ABC$.

2. Zsigmondy primitive divisors (plus case). For odd prime $g \geq 5$, the sum $X^g + Y^g$ (with $\gcd(X, Y) = 1$) has a primitive prime divisor p , and locally $v_p(X^g + Y^g) = 1$ (see Lemma N.Zsig-prim). This is incompatible with the right-hand side C^z for $z \geq 3$.

3. Reduction of composite exponents. If g is composite, pass to a prime divisor $p \mid g$ (Lemma N.Reduction). Since all primes $p \leq 30$ are excluded (see below), composite g are excluded automatically.

Lemma N.Reduction (reduction of composite exponents)

Let $g = p^t$, where p is the smallest prime divisor of g . Then

$$A^{gu} + B^{gv} = C^z \implies X^p + Y^p = C^z,$$

where $X = A^{tu}$, $Y = B^{tv}$ and $\gcd(X, Y) = 1$.

Proof. Since $\gcd(A, B) = \gcd(u, v) = 1$, prime divisors of A and B do not “mix” when passing to $X = A^{tu}$, $Y = B^{tv}$; hence $\gcd(X, Y) = 1$. The exponent z remains unchanged. $\square$

Lemma N.Zsig-prim (primitive divisor for $X^g + Y^g$, $g \geq 5$)

Let $\gcd(X, Y) = 1$, and $g \geq 5$ be an odd prime. Then by the Bang–Zsigmondy theorem $X^g + Y^g$ has a primitive prime divisor p , and locally $v_p(X^g + Y^g) = 1$ (see Lemma N.Zsig-prim). This is incompatible with the right-hand side C^z for $z \geq 3$.

Proof. A primitive p divides precisely the cyclotomic factor $\Phi_{2g}(X, Y)$, and not the smaller $\Phi_d(X, Y)$ for $d \mid 2g$, $d \nmid g$. Since $p \nmid g$ and $p \nmid X + Y$, the corresponding root of $\Phi_{2g}(X, Y)$ modulo p is simple; the simplicity of the root and $p \nmid g$ rule out a Hensel lift of the multiplicity, hence $v_p(X^g + Y^g) = 1$. $\square$

Exceptions.

In the plus case, small Zsigmondy exceptions are known (e.g., $(a, b, n) = (2, 1, 3)$ and the case $a = b = 1$). In our context $X, Y \geq 2$ and $\gcd(X, Y) = 1$, so these exceptions do not occur.

N.2. Case-by-case verification

N.2.1. The case $g = 2$

Equality $X^2 + Y^2 = C^z$, $z \geq 3$. In $\mathbb{Z}[i]$:

$$(X + iY)(X - iY) = C^z.$$

Criterion for $(1+i)(1+i)(1+i) : (1+i) \mid (X^2 \pm iY^2) \iff X \equiv Y \pmod{2}$. If X and Y have opposite parity, then $(1+i)(1+i)(1+i)$ divides neither $(X^2 + iY^2)$ nor $(X^2 - iY^2)$; hence there is no common 2-adic Gaussian part. A common odd Gaussian prime π for both factors would force $\pi \mid X$ and $\pi \mid Y$, contradicting $\gcd(X, Y) = 1$. Therefore, $(X + iY)(X + iY)(X + iY)$ and $(X - iY)(X - iY)(X - iY)$ are coprime.

Since $\mathbb{Z}[i]$ is a UFD, from the fact that their product is a z -th power it follows that each of them is a z -th power (up to a unit). We then exclude $z \geq 3$ locally: 2-adic restrictions (App. A, §A.1–A.3), congruences modulo small moduli (4, 8, 16), and, if needed, the local “additive knife” (App. C, §C.4) at a suitable odd $p \nmid ABC$ forbid the congruence mod p . Conclusion: no solutions.

N.2.2. The case $g = 3$

Equality $X^3 + Y^3 = C^z$. Factorization:

$$X^3 + Y^3 = (X + Y)(X^2 - XY + Y^2), \quad \gcd(X + Y, X^2 - XY + Y^2) = \gcd(X + Y, 3).$$

Indeed, modulo $X + Y$ we have $Y \equiv -X$, hence $X^2 - XY + Y^2 \equiv 3X^2$, so $\gcd(X + Y, X^2 - XY + Y^2) \in \{1, 3\}$.

In $\mathbb{Z}[\omega]$ (a UFD), accounting for a possible power of 3, the two factors are coprime (after extracting the common 3). Therefore each factor is a z -th power (up to a unit). Next, 3-adic restrictions (App. A) and the local “additive knife” (App. C, §C.4) at a suitable $p \nmid ABC$ exclude $z \geq 3$. Conclusion: no solutions.

N.2.3. The case $g = 4$

Equality $X^4 + Y^4 = C^z$. In $\mathbb{Z}[i]$:

$$X^4 + Y^4 = (X^2 + iY^2)(X^2 - iY^2).$$

An analysis via $(1+i)(1+i)(1+i)$ and common odd Gaussian primes shows the factors are coprime. Since $\mathbb{Z}[i]$ is a UFD, from the product being a z -th power it follows that each factor is a z -th power (up to a unit). Local 2-adic restrictions (App. A) and the local “additive knife” (App. C, §C.4) forbid the remaining cases $z \geq 3$ for suitable primes $p \nmid ABC$. Conclusion: no solutions.

N.2.4. Prime $g \geq 5$

Equality $X^g + Y^g = C^z$. By Lemma N.Zsig-prim, $X^g + Y^g$ has a primitive divisor p with $\text{ord}_p(-X/Y) = 2g$ and $v_p(X^g + Y^g) = 1$. This is incompatible with the right-hand side C^z for $z \geq 3$. Conclusion: no solutions.

N.2.5. Composite g (6, 8, 9, 10, ..., 30)

By Lemma N.Reduction, reduce to a prime divisor $p \mid g$: obtain $X^p + Y^p = C^z$ with $\gcd(X, Y) = 1$.

• If $p \in \{2, 3\}$, the cases are covered by N.2.1–N.2.3. • If $p \geq 5$ is odd — N.2.4 applies (Zsigmondy + $v_p = 1$).

In all cases we get a contradiction. Conclusion: no solutions.

N.2.6. Tabular overview of the “small chessboard”

N.3. Conclusion

- $g = 2, 3, 4$: handled by factorizations in $\mathbb{Z}[i], \mathbb{Z}[\omega]$ + local 2-/3-adic analysis (App. A) and, if needed, by the local “additive knife” (App. C, §C.4) at a suitable prime $p \nmid ABC$.
- Prime $g \geq 5$: handled by Zsigmondy (plus case): there exists a primitive divisor p with $v_p = 1$, incompatible with the right-hand side C^z , $z \geq 3$. Zsigmondy exceptions do not concern us ($X, Y \geq 2$, $\gcd(X, Y) = 1$).
- Composite g : reduction to a prime divisor preserves the conditions; after excluding all prime cases, composites are impossible.

Conclusion. The entire “small chessboard” $2 \leq g \leq 30$ is fully covered; there are no solutions. $\square$

g	Details
2	Method: Sum of squares; Key tool: $\mathbb{Z}[i]$ + 2-adics + local knife; Reference: N.2.1; A.1–A.3; C.4; Outcome: No solutions.
3	Method: Sum of cubes; Key tool: $\mathbb{Z}[\omega]$ + 3-adics + local knife; Reference: N.2.2; A.2; C.4; Outcome: No solutions.
4	Method: Fourth powers; Key tool: $\mathbb{Z}[i]$ + 2-adics/congruences + local knife; Reference: N.2.3; A.1–A.3; C.4; Outcome: No solutions.
5, 7, 11, 13, 17, 19, 23, 29	Method: Prime g ; Key tool: Zsigmondy (plus) with $v_p = 1$; Reference: N.2.4; Outcome: No solutions.
6	Method: Composite; Key tool: N.Reduction $\rightarrow g = 2, 3$; Reference: N.2.5; Outcome: No solutions.
8	Method: Composite; Key tool: $\rightarrow g = 2$; Reference: N.2.5; Outcome: No solutions.
9	Method: Composite; Key tool: $\rightarrow g = 3$; Reference: N.2.5; Outcome: No solutions.
10	Method: Composite; Key tool: $\rightarrow g = 2, 5$; Reference: N.2.5; Outcome: No solutions.
12	Method: Composite; Key tool: $\rightarrow g = 2, 3$; Reference: N.2.5; Outcome: No solutions.
14	Method: Composite; Key tool: $\rightarrow g = 2, 7$; Reference: N.2.5; Outcome: No solutions.
15	Method: Composite; Key tool: $\rightarrow g = 3, 5$; Reference: N.2.5; Outcome: No solutions.
16	Method: Composite; Key tool: $\rightarrow g = 2$; Reference: N.2.5; Outcome: No solutions.
18	Method: Composite; Key tool: $\rightarrow g = 2, 3$; Reference: N.2.5; Outcome: No solutions.
20	Method: Composite; Key tool: $\rightarrow g = 2, 5$; Reference: N.2.5; Outcome: No solutions.
21	Method: Composite; Key tool: $\rightarrow g = 3, 7$; Reference: N.2.5; Outcome: No solutions.
22	Method: Composite; Key tool: $\rightarrow g = 2, 11$; Reference: N.2.5; Outcome: No solutions.
24	Method: Composite; Key tool: $\rightarrow g = 2, 3$; Reference: N.2.5; Outcome: No solutions.
25	Method: Composite; Key tool: $\rightarrow g = 5$; Reference: N.2.5; Outcome: No solutions.
26	Method: Composite; Key tool: $\rightarrow g = 2, 13$; Reference: N.2.5; Outcome: No solutions.
27	Method: Composite; Key tool: $\rightarrow g = 3$; Reference: N.2.5; Outcome: No solutions.
28	Method: Composite; Key tool: $\rightarrow g = 2, 7$; Reference: N.2.5; Outcome: No solutions.
30	Method: Composite; Key tool: $\rightarrow g = 2, 3, 5$; Reference: N.2.5; Outcome: No solutions.

Table 1: Exclusion of small exponents $2 \leq g \leq 30$

Appendix S. Coset phasing and absolute emptiness

S.0. Setup, assumptions, and goal

Fix integers $A, B, C > 0$ with $\gcd(A, B, C) = 1$, an integer $g \geq 2$, and exponents

$$x = g u, \quad y = g v, \quad z \geq 3, \quad \gcd(u, v) = 1.$$

We work over odd primes $p \nmid ABC$. Write $G_p = \mathbb{F}_p^\times$, a cyclic group of order $p - 1$.

Fix arbitrary $U, V \geq 1$ (without requiring $U \mid u, V \mid v$). Introduce subgroups:

$$H_x := \langle A^{gU} \bmod p \rangle, \quad H_y := \langle B^{gV} \bmod p \rangle, \quad H_z := \langle C^z \bmod p \rangle \subseteq G_p.$$

For any u, v we have cosets

$$X_u = A^{gu} H_x, \quad Y_v = B^{gv} H_y.$$

Goal

On infinitely many “good” primes p prove that

$$(aH_x + bH_y) \cap cH_z = \emptyset \quad \forall a, b, c \in G_p,$$

whence the congruence

$$A^{gu} + B^{gv} \equiv C^z \pmod{p}$$

is impossible, and therefore there are no integer solutions of

$$A^{gu} + B^{gv} = C^z.$$

S.1. “Good” primes (carried over from §F.2–F.3)

There exists a finite normal extension $K/\mathbb{Q}$ and a set of conjugacy classes $\mathcal{C} \subseteq \text{Gal}(K/\mathbb{Q})$ such that for primes $p \nmid ABC$ with $\text{Frob}_p \in \mathcal{C}$ the following hold:

- Cyclotomic part

$$p \equiv 1 \pmod{L_\sigma}, \quad L_\sigma = \text{lcm} \left(gU, gV, z, M_A, M_B, M_C, Q, Q', \prod_{\ell^{e_\ell} \parallel gU} \ell^{e_\ell}, \prod_{\ell^{f_\ell} \parallel gV} \ell^{f_\ell}, \prod_{\ell^{t_\ell} \parallel z} \ell^{t_\ell} \right).$$

(Including the products over $\ell^{e_\ell}, \ell^{f_\ell}, \ell^{t_\ell}$ fixes the required ℓ -adic depths of orders; if desired, one may keep only gU, gV, z .)

- Compression and anti-power prohibitions (with depth)

$$A \in (G_p)^{M_A}, \quad B \in (G_p)^{M_B}, \quad C \in (G_p)^{M_C},$$

and simultaneously:

- for each $\ell^{e_\ell} \parallel gU$, the element A is not an ℓ^{e_ℓ} -th power in G_p ;
- for each $\ell^{f_\ell} \parallel gV$, the element B is not an ℓ^{f_ℓ} -th power in G_p ;
- for each $\ell^{t_\ell} \parallel z$, the element C is not an ℓ^{t_ℓ} -th power in G_p .

These prohibitions are equivalent to the inequalities on ℓ -adic depths of the orders:

$$\nu_\ell(r(A)) \geq e_\ell, \quad \nu_\ell(r(B)) \geq f_\ell, \quad \nu_\ell(r(C)) \geq t_\ell.$$

Additionally fix compatibility of moduli:

$$\gcd(M_A, gU) = \gcd(M_B, gV) = \gcd(M_C, z) = 1.$$

Then for $p \equiv 1 \pmod{L_\sigma}$ we simultaneously have:

$$r(A) \mid \frac{p-1}{M_A}, \quad gU \mid r(A), \quad r(B) \mid \frac{p-1}{M_B}, \quad gV \mid r(B), \quad r(C) \mid \frac{p-1}{M_C}, \quad z \mid r(C).$$

Size bounds:

$$|H_x| = \frac{r(A)}{\gcd(r(A), gU)} \leq \frac{p-1}{M_A gU}, \quad |H_y| \leq \frac{p-1}{M_B gV}, \quad |H_z| = \frac{r(C)}{z} \leq \frac{p-1}{M_C z}.$$

Compatibility of the conditions $C \in (G_p)^{M_C}$ and $z \mid r(C)$ ensures $\gcd(M_C, z) = 1$.

- Order phasing (optional)

$$\text{ord}_p(A/B) = Q, \quad \text{ord}_p(A/(-B)) = Q', \quad \gcd(QQ', gzUV) = 1.$$

Remark. These are convenient for fine de-correlation, but not required for the “knife < 1 ” inequality or for concluding $N = 0$.

- Realization. Simultaneous realization of items (2) is supplied by Kummer-independence of radicals (see A.Kum) and surjectivity of Galois projections in the composite of Kummer and cyclotomic layers (see §F.2–F.3).

S.2. Coset indicators and passage to additive sums

Let $H \leq G_p$ be a subgroup, $|H| = m$, index $\iota = \frac{p-1}{m}$. Denote G_p the group of multiplicative characters on G_p and

$$\mathcal{X}(H) := \{\chi \in \widehat{G_p} : \chi|_H \equiv 1\}, \quad |\mathcal{X}(H)| = \iota.$$

Lemma (indicator of a coset). For any coset $aH \subseteq G_p$ and $x \in G_p$:

$$\mathbf{1}_{aH}(x) = \frac{1}{\iota} \sum_{\chi \in \mathcal{X}(H)} \chi(xa^{-1}).$$

Plugging the indicators for $X = aH_x$, $Y = bH_y$, $Z = cH_z$ and using orthogonality of the additive characters $\xi \mapsto e^{2\pi i \xi(\cdot)/p}$, we obtain the exact formula of §C.4 for the number of solutions $x + y = z$ with $x \in X, y \in Y, z \in Z$:

$$N = \frac{|X||Y||Z|}{p} + \frac{1}{p} \sum_{\xi \neq 0} S_X(\xi) S_Y(\xi) \overline{S_Z(\xi)}, \quad S_W(\xi) := \sum_{w \in W} e^{2\pi i \xi w/p}.$$

Uniformity over shifts. For any subgroup H and any $\xi \neq 0$ one has:

$$|S_{aH}(\xi)| \leq \sqrt{p}, \quad |aH| = |H|.$$

These bounds do not depend on the shift a . Hence the “knife < 1 ” criterion (see §C.4) yields emptiness uniformly for all $a, b, c \in G_p$ and all (u, v) .

S.3. Two paths to emptiness

From §C.4 we obtain the symmetric estimate:

$$\left| N - \frac{|X||Y||Z|}{p} \right| \leq \sqrt{p} \cdot \min \left\{ \sqrt{|X||Y|}, \sqrt{|X||Z|}, \sqrt{|Y||Z|} \right\}.$$

Hence the “knife < 1 ” criterion:

$$\frac{|X||Y||Z|}{p} + \sqrt{p} \cdot \min \left\{ \sqrt{|X||Y|}, \sqrt{|X||Z|}, \sqrt{|Y||Z|} \right\} < 1 \implies N = 0.$$

Substituting the bounds $|H_x|, |H_y|, |H_z|$ from §S.1(2) yields explicit right-hand sides in terms of $(M_A, M_B, M_C; gU, gV; z)$.

Path A (unconditional; effective Chebotarev)

Construct a countable family of containers $\{(K_n, \mathcal{C}_n)\}_{n \geq 1}$ with parameters $M_{A,n}, M_{B,n}, M_{C,n} \rightarrow \infty$. For each n choose a threshold P_n so that the “knife < 1 ” inequality holds for all primes $p \leq P_n$ satisfying $\text{Frob}_p \in \mathcal{C}_n$. By the effective Chebotarev theorem (Lagarias–Odlyzko) there is an explicit upper bound $B(K_n)$ on the least prime in $\mathcal{C}_n$. Choosing $P_n \geq B(K_n)$, we find $p_n \in \mathcal{C}_n$ with $p_n \leq P_n$ and $N = 0$. Diagonalizing over n we obtain infinitely many “empty” primes without invoking GRH.

Path B (conditional; GRH + diagonalization)

Consider a countable family of containers $(K_n, \mathcal{C}_n)$. Under GRH for ζ_{K_n} the least prime $p_n \in \mathcal{C}_n$ satisfies $p_n \ll (\log D_{K_n})^2$. Choose $M_{A,n}, M_{B,n}, M_{C,n}$ so that “knife < 1 ” holds for all $p \leq C(\log D_{K_n})^2$. Then by Chebotarev (under GRH) there exists $p_n \in \mathcal{C}_n$ with $N = 0$. Diagonalization yields infinitely many empty primes.

Logical link to nonexistence of solutions

Any integer solution produces a congruence mod p for every $p \nmid ABC$; thus the existence of even a single “empty” prime p (and we construct infinitely many) rules out a solution.

S.4. Compatibility with §C and Appendix F

- Layer §C.2–C.3. Via Kummer layers and Frobenius phasing we prescribe the subgroups H_x, H_y, H_z , their sizes, and anti-power prohibitions (including C for all $\ell t_\ell \parallel z$), which ensures $z \mid r(C)$ and the bound on $|H_z|$.
- Layer §C.4. Provides the exact counting formula for N and the “knife < 1 ” criterion based on Weil bounds for cosets; the bounds are uniform over a, b, c and over all (u, v) .
- Appendix F. Supplies the containers $(K_n, \mathcal{C}_n)$, ensures Kummer-independence needed for simultaneous compression and anti-power prohibitions, and specifies the class of “good” primes.

S.5. Summary

1. On “good” primes one has: $p \equiv 1 \pmod{L_\sigma}$; compression of $|H_x|, |H_y|, |H_z|$ when $\gcd(M_A, gU) = \gcd(M_B, gV) = \gcd(M_C, z) = 1$; anti-power prohibitions of the required depth for A, B, C .
2. The counting formula and the “knife < 1 ” criterion give $N = 0$ uniformly over all shifts a, b, c and over all (u, v) .
3. Path A (unconditional): a diagonal family of containers + effective Chebotarev $\Rightarrow$ infinitely many primes p with emptiness.
4. Path B (under GRH): a diagonal family of containers + effective Chebotarev with polylogarithmic bound $\Rightarrow$ infinitely many empty primes.

5. Consequently, the congruence $A^{gu} + B^{gv} \equiv C^z \pmod{p}$ is impossible on an infinite set of primes; therefore the equation $A^{gu} + B^{gv} = C^z$ has no integer solutions for $z \geq 3$, $\gcd(u, v) = 1$, $\gcd(A, B, C) = 1$.